Asset Pricing When Trading is Entertainment

Jiang Luo* and Avanidhar Subrahmanyam**

*Nanyang Business School, Nanyang Technological University, Singapore.

**Corresponding author. Address: Anderson Graduate School of Management, University of California at Los Angeles, Los Angeles, CA 90095-1481, USA; Phone: (310) 825-5355; Fax: (310) 206-5455; email: subra@anderson.ucla.edu.

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We analyze a model where agents derive direct utility from the act of trading. The resulting equilibrium shows that trading for entertainment creates an excessive tendency to buy low and sell high, and causes attenuation of covariance risk pricing and return volatility. The model also implies a negative relation between volume and future returns, consistent with the empirical evidence. Further, if agents derive greater utility from trading more volatile stocks, our model is consistent with the “volatility anomaly” wherein more volatile stocks earn lower average returns in the cross-section, although the risk premium on the market portfolio is positive. Additionally, we show that increased attractiveness of alternative opportunities to participate in risky activities reduces trading volume in equities. If agents’ utility from trading increases when they make positive profits in earlier rounds, this leads to “bubbles,” i.e., disproportionate jumps in asset returns as a function of past prices.
1 Introduction

“The game of investing is intolerably boring and over-exacting to any one
who is entirely exempt from the gambling instinct; whilst he who has it must
pay to this propensity the appropriate toll.”

– Keynes (1936)

Why do agents trade? In the neoclassical paradigm, trade occurs to rebalance portfolios
according to the risk-return tradeoff, either upon the change of market values of securities,
or due to a change in preferences, or due to the receipt of new information.¹ However,
volume in financial markets appears to be too large to be explained by such considerations
alone. Milgrom and Stokey (1982) imply that there should be no trade among investors
with only speculative motives for trading. Barber and Odean (2000), however, report
extremely high levels of trading; specifically, they document an average annual turnover
of 75% by customers with accounts at a large discount brokerage firm. According to
the NYSE website,² annual turnover on the NYSE has ranged between 60% and 100%
of shares outstanding over the past ten years. De Bondt and Thaler (1995) (p. 392)
note that “the high trading volume observed in financial markets is perhaps the single
most embarrassing fact to the standard finance paradigm.” Indeed, Tkac (1999) shows
that real-world volume exceeds that indicated by rational portfolio-rebalancing for a vast
majority of traded stocks. Motivated by the generally high levels of volume in financial
markets, Black (1986) (p. 531) mentions the need to “introduce direct utility of trading”
to explain volume.³

¹See Grossman and Stiglitz (1980), Kyle (1985) Grundy and McNichols (1989), Foster and
Viswanathan (1993), and Wang (1994). Trading is induced by differences of opinion in Harrison and
³A recent article in the Wall Street Journal (“Lets Be Honest: Are You an Investor or a Speculator?”

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The goal of our paper is to consider a financial market equilibrium where agents derive utility from the act of trading. In much of our work, information is symmetric and there are multiple assets, each traded by agents who possess the standard exponential-normal utility function over wealth but some of whom get additional utility from trading. We do not model the origins of this utility; it could possibly emanate from the thrill of seeing position values fluctuate. The notion that agents may gamble for pleasure is well-established in literature (see, for example, Coventry and Brown (1993) or Kuley and Jacobs (1988)), and Markiewicz and Weber (2013) show that gambling tendencies also create excessive stock trading. Motivated by these observations, we simply consider the nature of equilibrium in financial markets when trading is a consumption good.\footnote{Barrot, Kaniel, and Sraer (2016) and Kaniel, Saar, and Titman (2008) demonstrate that individual investors do act as liquidity providers in financial markets.}

To attain tractability, we assume that the direct utility is convex (quadratic) in the amount of the traded asset. We find that when agents derive utility from trading, they have an excessive tendency to “buy low and sell high.” The intuition is that the normal levels of trade to capture risk premia are magnified owing to the additional utility derived from trading. Interestingly, such agents result in lower asset volatility. This is because they act as de facto liquidity providers in equilibrium.\footnote{An issue is whether the mass of traders who trade for entertainment is sufficiently large to affect prices. While this is an empirical question, Barber, Odean, and Zhu (2009) show that retail traders, (who are more likely to trade for enjoyment than professionals) do have an impact on financial market prices.}

We also find that beta pricing is obscured when agents derive direct utility from trading. Specifically, expected returns are linear in beta, but the coefficient on beta attenuates as the direct utility from trading increases. This result accords with the notion that it is generally hard to find evidence of covariance risk pricing.

by Jason Zweig, December 9, 2016) quotes “Barry Metzger, a senior vice president for trading services at Charles Schwab Corp.,” as saying that some active traders “believe that trading is fun and a part of who they are,” indicating that they view trading as a consumption good.
in equity markets (see, for example, Fama and French (1992) and Haugen and Baker (1996)). The intuition in our setting is that the compensation for risk demanded in equilibrium declines as the direct utility from trading rises.

It is reasonable to suppose that agents may derive greater utility from some stocks relative to others. The two main characteristics proposed by Kumar (2009) for stocks that are attractive to individual investors are high (positive) skewness, and high volatility. Since our model has normally distributed payoffs it unfortunately cannot speak to skewness preference. We instead consider the assumption that agents obtain more utility from stocks with more volatile payoffs. We find that under reasonable conditions, such stocks get “overvalued” and earn negative future returns on a risk-adjusted basis. Since high payoff volatility, under reasonable conditions, also corresponds to high beta and high idiosyncratic volatility (as we show), our analysis is consistent with the empirical result of Ang, Hodrick, Xing, and Zhang (2006) that idiosyncratic volatility negatively forecasts asset returns.

An interesting set of stylized facts in finance is that while volatility is negatively priced in the cross-section, on aggregate, the risk premium on equities is positive (Haugen and Baker (2010), Mehra and Prescott (1985)). We show that our analysis accords with the “low volatility” anomaly wherein low risk stocks earn higher average returns than high risk stocks (as shown empirically in Baker and Haugen (2012)). At the same time, however, the market portfolio commands a positive risk premium because our agents are risk averse. Thus, our analysis is simultaneously consistent with positive risk pricing in the aggregate, but negative pricing of volatility in the cross-section.

When agents derive direct utility from trading, they create additional volume, which accords with levels of volume greater than normally expected levels from neoclassical
models (Tkac (1999)). Further, when trading volume is high in a stock, it is associated with agents desiring greater utility from trading that stock. However, stocks from which agents derive greater utility from trading also tend to become more overpriced. Thus, our model is consistent with the negative cross-sectional relation between volume and returns documented in Datar, Naik, and Radcliffe (1998) or Brennan, Chordia, and Subrahmanyam (1998). Additionally, we show that the relative attractiveness of alternative opportunities to participate in risky activities (such as casino gambling and lotteries) affects equilibrium trading volume in equities. Specifically, when alternative opportunities are more attractive, volume in equities is lower. This is consistent with Gao and Lin (2015), who show that equity trading volume in Taiwan decreases as the total jackpot of a major statewide lottery increases, indicating that stock trading acts as an alternative outlet for gambling beyond lotteries.

Previous literature argues that the pleasure from gambling rises when the outcome of a previous gamble is positive (e.g., Coventry and Constable (1999); Thaler and Johnson (1990)). Accordingly, we extend our model to a dynamic setting where agents derive more utility from trading if previous rounds of trade have been profitable. We find that the impact of a modestly positive piece of news can be greatly magnified. Specifically, as the news crosses a threshold, it causes a discontinuous jump in the mass of agents who trade for entertainment purposes, which results, in turn, in a greatly enhanced response of asset prices to the news. This suggests that relatively modest favorable price moves can create “bubbles” in asset prices, wherein later price moves represent substantial overreactions.

6In a complementary and important view Odean (1998) and Statman, Thorley, and Vorkink (2006) consider the notion that overconfidence creates excessive trading volume. A distinguishing feature of our setting and that of these papers is that overconfidence increases price volatility whereas in our setting agents who derive utility from trading act as de facto market makers and thus reduce volatility. In Daniel, Hirshleifer, and Subrahmanyam (2001) overconfidence attenuates beta pricing, just like in our paper. However, overconfidence is unlikely to explain phenomena such as negative pricing of risk, because such theory does not clearly predict that overconfident agents should be more attracted to risky assets.
to the initial price move, followed by subsequent corrections. Thus, in our setting, it is the “emotional excitement” caused by the initial price move which results in the bubble in asset prices, which accords with the experimental observation that excitement fuels bubbles (Andrade, Odean, and Lin (2015)).

Our analysis suggests untested implications. Specifically, we argue that stocks that are heavily traded and held by retail investors (who are more likely to trade for entertainment) should exhibit greater trading volume and lower volatility, and less evidence of covariance risk pricing. These stocks should also exhibit nonlinear responses to positive news. Our analysis also suggests that covariance risk pricing should be less visible in countries or economies where retail investors form a bigger fraction of the trading population.

The idea that agents may trade for purposes of deriving enjoyment from trading is not new; but explicit theoretical modeling of this notion does not yet appear in the literature. For example, Dorn and Sengmueller (2009) argue that stock market trading provides direct utility to agents in the form of entertainment. Specifically, they show that agents who state that they derive “enjoyment” from trading turn over their portfolios to a greater degree than other investors. In a survey of retail investors, Dhar and Goetzmann (2006) state that more than 25% of investors view stock market investing as a hobby. Grinblatt and Keloharju (2009) argue that sensation seeking personalities may obtain a thrill from the act of trading but again, do not explicitly model such investors. Gao and Lin (2015) show that equity trading volume in Taiwan decreases as the total jackpot of a major statewide lottery increases, indicating that stock trading acts as an alternative outlet for gambling beyond lotteries. Barberis and Xiong (2012) model the important insight that realizing gains conveys pleasure to agents, but they do not consider a setting where agents derive utility from trading. Our paper fills this void in the literature by

7Our paper is complementary to Friedman and Heinle (2016) and Luo and Subrahmanyam (2016), which consider a setting where agents derive direct utility or disutility from owning certain types of stocks.
explicitly considering agents for whom trading is a consumption good.

This paper is organized as follows. Section 2 presents the basic model, while compares economies with and without agents who receive direct utility from trading. Section 4 endogenizes investors’ participation in the stock market to gamble. Section 5 presents a dynamic extension, and Section 6 concludes. All proofs of propositions and corollaries, unless otherwise stated, appear in Appendix A, while Appendix B presents some ancillary derivations.

2 The Model

There are two dates, 0 and 1, and \( K + N \) risky securities. At date 1, these securities pay liquidating dividends of \( V = (V_1, \ldots, V_{K+N})' \), which follows a multivariate normal distribution. At date 0, investors trade these securities. The per capita supplies of these securities also follow a multivariate normal distribution. Security prices are indicated by \( P = (P_1, \ldots, P_{K+N})' \) and are determined in equilibrium. There is also a riskless asset, the price and return of which are normalized to unity.

There are two types of agents. First, there is a mass \( \rho \) of regular utility-maximizing agents. Second, there is a mass \( 1 - \rho \) of agents we call “\( G \) traders;” these achieve direct utility from trading risky securities. One can view the former class of agents as sophisticated investors (possibly institutions), and at least part of the latter class as unsophisticated (e.g., individual) investors.

The utility there emanates from the signed position; for example, an environmentally conscious agent derives disutility from owning stocks (and utility from shorting stocks) in firms that heavily use coal. In contrast, direct utility in our model emanates from the unsigned quantity of trade. Further, Doran, Jiang, and Peterson (2012), Shefrin and Statman (2000), and Brunnermeier, Gollier, and Parker (2007) consider the notion that agents may express an affinity towards gambling by preferring high skewness assets. We instead consider a complementary setting where agents have standard risk averse preferences but derive additional utility from the act of trading an asset (which is normally distributed).
The \( i \)'th trader is endowed with \( X_i = (X_{i1}, \ldots, X_{i,K+N})' \) units of risky securities and \( \bar{M}_i \) units of risk free asset. His wealth levels at Dates 0 and 1 are respectively given by

\[
W_{i0} = \bar{M}_i + \bar{X}_i' P, \\
W_{i1} = W_{i0} + X_i'(V - P),
\]

where \( X_i = (X_{i1}, \ldots, X_{i,K+N})' \) is the quantity of risky securities he holds after the trading is complete.

The utility function of the \( i \)'th regular (non-\( G \)) trader is the standard exponential one:

\[
U(W_{i1}) = -\exp(-\gamma W_{i1}),
\]

with \( \gamma > 0 \). Based on the normality assumption of our model, he chooses \( X_i \) to maximize

\[
E[U(W_{i1})] = -\exp\left[ -\gamma W_{i0} - \gamma \left( X_i'E(V - P) - 0.5\gamma X_i'\text{Var}(V)X_i \right) \right].
\]

The first order condition (f.o.c.) with respect to (w.r.t.) \( X_i \) implies that his demand can be expressed as:

\[
X_{NG}(P) = \frac{1}{\gamma} \text{Var}(V)^{-1}(E(V) - P).
\]

However, the \( i \)'th \( G \) trader has the utility function:

\[
U_G(W_{i1}, X_i) = -\exp(-\gamma W_{i1})\exp(-0.5X_i'GX_i),
\]

where \( G \) is a diagonal, positive definite matrix. The above utility function captures the notion that the bigger the quantity of trade (in absolute terms), the bigger is the utility derived from trading. It may be argued that increasing the scale of the transaction increases the “thrill” derived from this larger scale of “gambling” in financial markets (Dorn and Sengmueller (2009)). The \( G \) trader chooses \( X_i \) to maximize

\[
E[U_G(W_{i1}, X_i)]
= -\exp\left[ -\gamma W_{i0} - \gamma \left( X_i'E(V - P) - 0.5\gamma X_i'\text{Var}(V)X_i + 0.5X_i'GX_i/\gamma \right) \right].
\]
The f.o.c. w.r.t. $X_i$ implies that his demand can be expressed as:

$$X_G(P) = \frac{1}{\gamma} (\text{Var}(V) - G/\gamma^2)^{-1}(E(V) - P).$$

The s.o.c. holds under the assumption that $\text{Var}(V) - G/\gamma^2$ is a positive definite matrix.

As can be seen, the “numerator” in the above demand is the same as for non-$G$ traders. However the position is larger per unit gain relative to non-$G$ traders, i.e., the $G$ traders take more aggressive positions relative to traditional utility maximizers. As the utility of trading increases, the position vector explodes, and beyond a certain level of $G$, there is no interior optimum. The scale of the position taken per unit expected price appreciation increases in $G$, which governs how much additional utility is derived from trading.

### 2.1 Risky Security Payoffs–The Factor Structure

We now explicitly model security payoffs as a factor structure to analyze how volume, volatility, and the pricing of covariance risk are affected by the presence of $G$ traders. In what follows, unless otherwise specified, a generic random variable, $\tilde{\eta}$, follows a normal distribution with mean zero and variance $\nu_{\eta}$.

The payoff of the $j$’th risky security takes a factor expression:

$$V_j = \bar{V}_j + \sum_{k=1}^{K} (\beta_{jk} \tilde{f}_k) + \tilde{\epsilon}_j$$

All $\tilde{f}$’s and $\tilde{\epsilon}$’s follow independent normal distributions.

### 2.2 An Equivalent Maximization Problem

As in Daniel, Hirshleifer, and Subrahmanyam (2001), we use the risky securities to construct portfolios mimicking the $K$ factors and $N$ residuals. We refer to these portfolios as the basic securities. Use $\tilde{\theta}_j$, $j = 1, ..., K + N$, to denote the payoffs of the basic securities.
Specifically, the first $K$ $\tilde{\theta}$’s indicate the payoffs of the $K$ factors, that is, $\tilde{\theta}_j = \tilde{f}_j$ for $j = 1, ..., K$. The next $N$ $\tilde{\theta}$’s indicate the payoffs of the $N$ residuals, that is, $\tilde{\theta}_{K+j} = \tilde{\epsilon}_j$ for $j = 1, ..., N$.

The per capita supply of the $j$’th basic security is indicated by $\bar{\xi}_j + \bar{z}_j$, where $\bar{\xi}_j$ is a constant. Henceforth, we will assume that the mean supply of securities is positive (i.e. $\bar{\xi}_j > 0 \ \forall j$). The supply noise is not necessary for most of our main results (except those on price volatility). To simplify our analysis, we assume that the variance $\nu_{z_j}$ is sufficiently small. Specifically, we assume that

$$\nu_{z_j} < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2).$$

This condition facilitates the derivation of the results because it ensures that the $G$ traders’ penchant for aggressively “buying low and selling high,” which drives many of our results, is not too adversely affected by excessive supply noise. The assumption is reasonable because we would not expect uncertainty in stock issuance and buyback activity to be unduly large in general.

We denote the utility from trading the $j$’th basic security as $G_j$. This implies that the $G$ traders’ utility from trading is not identical across stocks, which captures the notion that there are many stock-specific properties that may appeal to a $G$ trader; for example, brand appeal, volatility, industry sector, and so on. Rather than model these attributes, we simply let the direct utility from trading vary in the cross-section. We do, however, consider a scenario in Section 2.6 where $G_j$ depends positively on the volatility of the stock.

Note that $\tilde{\theta}_j \sim N(0, \nu_{\theta_j})$. Each non-$G$ and $G$ trader’s demands for the $j$’th basic security are given by

$$X_{NG,j}(P_j) = \frac{E(\tilde{\theta}_j) - P_j}{\gamma \text{Var}(\tilde{\theta}_j)} = \frac{-P_j}{\gamma \nu_{\theta_j}},$$

(3)
\[
X_{G,j}(P_j) = \frac{E(\tilde{\theta}_j) - P_j}{\gamma(\text{Var}(\theta_j) - G_j/\gamma^2)} = \frac{-P_j}{\gamma(\nu_{\theta_j} - G_j/\gamma^2)}. \tag{4}
\]

The denominator in Eq. (4) needs to be positive for the second-order condition of the \( G \) traders to be satisfied. The intuition is that the effect of \( G_j \) alone is to induce \( G \)-traders to want to trade infinite quantities. This tendency, however, is tempered by risk aversion and the volatility of the asset’s final payoff. Henceforth, we will assume that the condition for an interior optimum is indeed satisfied, i.e., that \( G_j < \gamma^2\nu_{\theta_j} \).

The market clearing condition requires
\[
\tilde{\xi}_j + \tilde{z}_j = \rho X_{NG,j}(P_j) + (1 - \rho)X_{G,j}(P_j),
\]
from which we derive the prices and returns (i.e., price changes) as presented in the following proposition.

**Proposition 1** The price and the return of the \( j \)’th basic security are given by
\[
P_j = -\gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j),
\]
\[
\tilde{R}_j = \tilde{\theta}_j - P_j = \tilde{\theta}_j + \gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j),
\]
where \( a_j(G_j) = \frac{1}{\frac{\rho}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2}}. \)

The term \( a_j(G_j) \) represents the effect of \( G \) traders on the price and required return. By being willing to trade more stock for a given market price, the \( G \) traders assist the non-\( G \) traders in absorbing the supply of the risky asset.

The corollary below follows immediately from the expression for \( a_j(G_j) \) in the above proposition:

**Corollary 1** (i) \( a_j(G_j) \) decreases in \( G_j \), with \( a_j(0) = \nu_{\theta_j} \).
(ii) \( a_j(G_j) \) increases in \( \rho \).

Thus, the price of the \( j \)'th basic security is a risk premium, and this premium decreases in \( G_j \), the direct utility of trading the \( j \)'th basic security and in \( 1 - \rho \), the proportion of \( G \) traders. As the utility from trading grows without bound, the risk premium on the security decreases. Hence, agents who derive greater utility from trading reduce risk premia and required returns on the security.

It is instructive to calculate the expected profits of the \( G \) traders. From Eqs. (3) and (4) and Proposition 1, the expected profits earned by a non-\( G \) and \( G \) trader, respectively, in the \( j \)'th basic security, are given by

\[
E \Pi_{NG,j} = E \left[ X_{NG,j}(P_j) \tilde{R}_j \right] = E \left[ \frac{\gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j)}{\gamma \nu_{\theta_j}} (\tilde{\theta}_j + \gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j)) \right]
\]

\[
= \frac{\gamma a_j(G_j)^2}{\nu_{\theta_j}} (\tilde{\xi}_j^2 + \nu_{\xi_j}),
\]

\[
E \Pi_{G,j} = E \left[ X_{G,j}(P_j) \tilde{R}_j \right] = E \left[ \frac{\gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j)}{\gamma (\nu_{\theta_j} - G_j/\gamma^2)} (\tilde{\theta}_j + \gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j)) \right]
\]

\[
= \frac{\gamma a_j(G_j)^2}{\nu_{\theta_j} - G_j/\gamma^2} (\tilde{\xi}_j^2 + \nu_{\xi_j}).
\]

Note that \( E \Pi_{G,j} > E \Pi_{NG,j} \). Thus, \( G \) traders earn a greater expected profit from trading the basic security than do non-\( G \) traders. This simply emanates from the notion that, in effect, they trade more aggressively to capture the risk premium.

Of course, it follows that \( G \) traders bear more risk than non-\( G \) traders. Indeed, it is easy to show that the expected utility of wealth is smaller for \( G \) traders relative to non-\( G \) traders, that is \( E \left[ -\exp(-\gamma W_i) \right] \) for the \( i \)'th \( G \) trader is less than \( E \left[ -\exp(-\gamma W_j) \right] \) for the \( j \)'th non-\( G \) trader.\(^8\) Our result is similar to that of Kyle and Wang (1997) who show that overconfidence acts as a commitment to trade aggressively in a strategic environment and thus results in greater expected profit for overconfident agents relative to that for rational

\(^8\)This result is obvious from the fact that \( G \) traders do not maximize \( E \left[ -\exp(-\gamma W_i) \right] \).
agents. In our setting, agents who obtain direct utility from trading obtain a greater expected profit than neoclassical agents because the increased position size of the former class of agents increases their average gain, relative to that for the latter class. Like De Long, Shleifer, Summers, and Waldmann (1991), who obtain a similar result on the expected profits of overconfident vs. rational traders, our result also challenges the notion that $G$ traders would not exist because they would persistently lose money in financial markets.

## 2.3 Volatility

It follows from Proposition 1 that the price and return volatilities of the $j$’th basic security are

\[
\text{Var}(P_j) = (\gamma a_j(G_j))^2 \nu_{z_j},
\]

\[
\text{Var}(\tilde{R}_j) = \nu_{\theta_j} + (\gamma a_j(G_j))^2 \nu_{z_j}.
\]

Express the return of the market portfolio as \( \tilde{R}_M = \sum_{j=1}^{K+N} (\xi_j + \tilde{z}_j) \tilde{R}_j \). We can compute the return volatility of the market portfolio. (The computation of this variance, which is tedious, is provided in Appendix B.)

\[
\text{Var}(\tilde{R}_M) = \sum_{j=1}^{K+N} \left[ (\nu_{\theta_j} + 4\gamma^2 a_j(G_j)^2 \nu_{z_j}) \xi_j^2 + \nu_{\theta_j} \nu_{z_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}^2 \right].
\] (6)

Corollary 1 then implies the following results on the volatilities.

**Corollary 2**  
(i) The individual security’s volatilities, \( \text{Var}(P_j) \) or \( \text{Var}(\tilde{R}_j) \), decrease in $G_j$ and increase in $\rho$.

(ii) The aggregate volatility of the market portfolio, \( \text{Var}(\tilde{R}_M) \), decreases in $G_j \, \forall j$ and increases in $\rho$. 
These volatilities decrease in $1 - \rho$, the mass of $G$ traders (and, of course, decrease in $G_j$). This is because $G$ traders behave like *de facto* liquidity providers. When prices increase (drop), they sell (buy) more relative to the non-$G$ traders.

An empirical proxy of $\rho$ is the percentage holdings of institutional investors. The time trend in past decades is that institutional holdings have increased (e.g., Chordia, Roll, and Subrahmanyam (2011)). At the same time, there is evidence that firms’ individual volatilities have increased (Campbell, Lettau, Malkiel, and Xu (2001)). Our evidence accords with these stylized facts.

Note that the results in Corollary 2 hold only if $G_j > 0$. If $\forall  j G_j = 0$, then $a_j(G_j) = \nu_{\theta_j}$ (see Corollary 1). The volatilities become $\text{Var}(P_j) = (\gamma \nu_{\theta_j})^2 \nu_{z_j}$, $\text{Var}(\tilde{R}_j) = \nu_{\theta_j} + (\gamma \nu_{\theta_j})^2 \nu_{z_j}$, and

$$\text{Var}(\tilde{R}_M) = \sum_{j=1}^{K+N} \left[ (\nu_{\theta_j} + 4 \gamma^2 \nu_{\theta_j}^2 \nu_{z_j}) \bar{\xi}_j^2 + \nu_{\theta_j} \nu_{z_j} + 2 \gamma^2 \nu_{\theta_j}^2 \nu_{z_j}^2 \right] ,$$

where the last equality follows from Eq. (6). These volatilities are independent of $\rho$.

### 2.4 The Pricing of Covariance Risk

We now turn to how the $G$ traders affect beta pricing. For this purpose, we fix $\rho > 0$ and vary $G_j$. We can compute the covariance between the returns of the $j$’th basic security and the market portfolio (the computation, which is tedious, is provided in Appendix B).

$$\text{Cov}(\tilde{R}_j, \tilde{R}_M) = \text{Cov}(\tilde{R}_j, \sum_{j=1}^{K+N} (\bar{\xi}_j + \bar{z}_j) \tilde{R}_j) = \text{Cov}(\tilde{R}_j, (\bar{\xi}_j + \bar{z}_j) \tilde{R}_j) = \nu_{\theta_j} \bar{\xi}_j + 2 \gamma^2 a_j(G_j)^2 \nu_{z_j} \bar{\xi}_j.$$  

\text{Cov}(\tilde{R}_j, \tilde{R}_M) = \sum_{j=1}^{K+N} \left[ (\nu_{\theta_j} + 4 \gamma^2 \nu_{\theta_j}^2 \nu_{z_j}) \bar{\xi}_j^2 + \nu_{\theta_j} \nu_{z_j} + 2 \gamma^2 \nu_{\theta_j}^2 \nu_{z_j}^2 \right] ,
Proposition 1 and Eq. (7) indicate that the expected return of the basic security can be expressed as:

$$E(\tilde{R}_j) = \gamma a_j(G_j)\tilde{\xi}_j = \frac{\gamma a_j(G_j)\Var(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2\nu_z} \beta_{jM}. \tag{8}$$

where $\beta_{jM} = \frac{\Cov(\tilde{R}_j, \tilde{R}_M)}{\Var(\tilde{R}_M)}$.

Let $\lambda_j \equiv \frac{\gamma a_j(G_j)\Var(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2\nu_z}$ denote the slope of the relation between $E(\tilde{R}_j)$ and $\beta_{jM}$. The following proposition describes the comparative static of $\lambda_j$ with respect to $G_j$.

**Proposition 2**

(i) Consider two basic securities, $j$ and $j'$, with $\nu_{\theta_j} = \nu_{\theta_{j'}}$, $\nu_z = \nu_{z'}$, but $G_j > G_{j'}$. Then, $\lambda_j < \lambda_{j'}$. The basic security with very large $G_j$ (i.e., $G_j/\gamma^2 \nearrow \nu_{\theta_j}$) has $\lambda_j \searrow 0$.

(ii) $\lambda_j$ increases in $\rho$.

This proposition suggests that high $G_j$ (or $\rho$) can lead to low $\lambda_j$ and, therefore, attenuate the predictive power of $\beta$’s. Particularly, $\lambda_j \searrow 0$ for stocks with very large $G_j$ (i.e., $G_j/\gamma^2 \nearrow \nu_{\theta_j}$). In this extreme case, $\beta$’s lose power in explaining stock return completely.

The basic intuition is that $G$-traders, via their tendency to “buy low and sell high” attenuate the pricing of risk (they reduce the equilibrium risk premium demanded by the non-$G$ traders). Gao and Lin (2015) show that stock trading and lottery participation are substitutes of some investors. An implication of our analysis is that when $G_j$ is high (e.g., during periods where substitute assets like lottery jackpots yield low payouts), then beta pricing will be weaker.

We now use Proposition 1 and Eq. (7) to express the $j$’th basic security’s expected return as

$$E(\tilde{R}_j) = \gamma a_j(G_j)\tilde{\xi}_j = \gamma \Cov(\tilde{R}_j, \tilde{R}_M) - \gamma \left[\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2\nu_z - a_j(G_j)\right] \tilde{\xi}_j$$

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\begin{equation}
\gamma \text{Var}(\tilde{R}_M)\beta_j - \gamma [\nu \theta_j + 2\gamma^2 a_j(G_j)^2 \nu z_j - a_j(G_j)]\tilde{\xi}_j, \tag{9}
\end{equation}

where the last item, \(-\gamma [\nu \theta_j + 2\gamma^2 a_j(G_j)^2 \nu z_j - a_j(G_j)]\tilde{\xi}_j\), is referred to as the $\beta$-adjusted expected return.

**Proposition 3**  
(i) The $\beta$-adjusted expected return of the $j$’th basic security, \(-\gamma [\nu \theta_j + 2\gamma^2 a_j(G_j)^2 \nu z_j - a_j(G_j)]\tilde{\xi}_j\), is negative.

(ii) Consider two basic securities, $j$ and $j'$, with $\nu \theta_j = \nu \theta_{j'}$, $\nu z_j = \nu z_{j'}$, $\tilde{\xi}_j = \tilde{\xi}_{j'}$, but $G_j > G_{j'}$. Then, the $\beta$-adjusted expected return of the $j$’th basic security is lower than that of the $j'$’th basic security.

(iii) The $\beta$-adjusted expected return of the $j$’th basic security increases in $\rho$.

The above proposition indicates that $G$-traders cause securities to on average be “over-priced” on a risk-adjusted basis. To see the intuition, first note that with positive average net supply, a high average price denotes a high risk premium on average, and prices on average decrease as they converge to fundamental values. However, since $G$ traders get direct utility from trading, when they absorb risky supplies, they are willing on average to pay more than rational investors for absorbing a given amount of supply, leading to greater “overpricing” than that naturally induced by risk premia.

### 2.5 Trading Volume

We now examine trading volume within our model. We aim to ascertain how trading volume is influenced by the presence of agents who derive direct utility from trading, and to investigate how volume might be associated with required returns on risky assets.

Let us assume that the initial endowment of the $j$’th basic security possessed by each agent equals the per capita mean supply $\tilde{\xi}_j + \tilde{z}_j$. It follows from Eq. (3) and Proposition 1
that the \(i\)'th non-\(G\) trader’s trade equals

\[
X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) \sim N \left[ \left( \frac{a_j(G_j)}{\nu_\theta_j} - 1 \right) \bar{\xi}_j, \left( \frac{a_j(G_j)}{\nu_\theta_j} - 1 \right)^2 \nu_z_j \right]. \tag{10}
\]

From Corollary 1, \(\left[ \frac{a_j(G_j)}{\nu_\theta_j} - 1 \right] \bar{\xi}_j < 0\). Therefore, non-\(G\) traders on average take a short position in the \(j\)'th basic security.

It follows from Eq. (4) and Proposition 1 that the \(i\)'th \(G\) trader’s trade equals

\[
X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) \sim N \left( \left( \frac{a_j(G_j)}{\nu_\theta_j} - \frac{G_j}{\gamma^2} - 1 \right) \bar{\xi}_j, \left( \frac{a_j(G_j)}{\nu_\theta_j} - \frac{G_j}{\gamma^2} - 1 \right)^2 \nu_z_j \right). \tag{11}
\]

It is straightforward to show that \(\left[ \frac{a_j(G_j)}{\nu_\theta_j} - \frac{G_j}{\gamma^2} - 1 \right] \bar{\xi}_j > 0\). Therefore, \(G\) traders on average take a long position in the \(j\)'th security.

The expected trading volume is given by half the sums of the expected absolute changes in each type of agent’s position via trading in the market for the \(j\)'th basic security. Using Eqs. (10) and (11), we can express the total expected trading volume in the basic security as

\[
T_j \equiv 0.5\rho E \left[ |X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)| \right] + 0.5(1 - \rho)E \left[ |X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)| \right]. \tag{12}
\]

We then have the following result.

**Corollary 3**  The expected trading volume, \(T_j\), increases in \(G_j\).

Corollary 3 indicates that stocks in which agents have a greater level of utility from trading exhibit greater trading volume. Intuitively, \(G\) traders trade such stocks intensely, which, in turn, implies that non-\(G\) traders take the opposite side of these positions in equilibrium, leading to high levels of volume. This result accords with Gao and Lin (2015), which suggests that volume in the stock market is high, lottery jackpots are low, and these are periods where we would expect the high \(G_j\) traders (gamblers) to
more actively participate in the stock market. Also, since the $\beta$-adjusted expected return is more negative, the greater is $G_j$ (Proposition 3), our analysis indicates that, ceteris paribus, stocks with high volume (i.e., high $G_j$ stocks) will earn low average returns on a risk-adjusted basis. This is consistent with the negative relation between trading volume and required returns documented, for example, in Datar, Naik, and Radcliffe (1998) and Brennan, Chordia, and Subrahmanyam (1998).\footnote{Based on Merton (1987) who argues that some (possibly, retail) investors might invest only in the most visible stocks, visibility (as measured by analyst following and brand visibility) might be a reasonable proxy for $G_j$. Our analysis suggests that such proxies will be associated with high volume and low average returns. In the next subsection, we consider another proxy for $G_j$, the volatility of the underlying asset’s cash flows.}

Based on Merton (1987) who argues that some (possibly, retail) investors might invest only in the most visible stocks, visibility (as measured by analyst following and brand visibility) might be a reasonable proxy for $G_j$. Our analysis suggests that such proxies will be associated with high volume and low average returns. In the next subsection, we consider another proxy for $G_j$, the volatility of the underlying asset’s cash flows.

### 2.6 $\beta$, Idiosyncratic Volatility, and Expected Returns

We now model a situation where the utility from trading an asset depends on its payoff volatility. This assumption is motivated from Kumar (2009), who shows that retail investors are more attracted to volatile companies.\footnote{In a complementary explanation, Baker and Stein (2004) argue that high volume implies high sentiment, and, under short-selling constraints, extreme optimism, that is reversed out the following month. In contrast, our rationale for the link between expected returns and volume does not rely on short-selling constraints.}

We model the dependence of $G_j$ on firms’ cash flow volatilities as follows. Denote $\bar{\nu}_\theta = \max(\nu_{\theta_j})$ and $\underline{\nu}_\theta = \min(\nu_{\theta_j})$, and let

$$G_j = G(\nu_{\theta_j}) = \gamma^2 \frac{\bar{\nu}_\theta}{\bar{\nu}_\theta - \underline{\nu}_\theta} (\nu_{\theta_j} - \underline{\nu}_\theta).$$

We assume that there is a sufficiently high mass of $G$ traders. Specifically,

$$\rho < \min\left(\frac{\nu_\theta}{\bar{\nu}_\theta}, 1 - \frac{\nu_\theta}{\bar{\nu}_\theta - \underline{\nu}_\theta}\right).$$

\footnote{The analysis of Kumar (2009), in turn, is derived from the notion that unsophisticated agents are more attracted to lotteries (Rubenstein, Scafidi, and Rubinstein (2002)), and lotteries demonstrate extremely high variance and high skewness. As our model assumes normal distributions, it cannot speak to skewness, of course, and considers volatility instead.}
We then have the following lemma:

**Lemma 1**  
(i) The expected return of the $j$'th basic security, $E(\tilde{R}_j) = \gamma a_j(G_j)\tilde{\xi}_j$ (see Eq. 8), decreases in $\nu_{\theta_j}$.

(ii) Ceteris paribus, the basic security with a higher $\nu_{\theta_j}$ also has a higher $\beta_{jM}$.

This lemma suggests that if there is a significant mass of $G$ traders (low $\rho$), then there can be a negative cross-sectional relation between $\beta$ and expected return. This result is consistent with negative beta pricing documented in some studies (Bali, Brown, Murray, and Tang (2016) (BBMT) and Frazzini and Pedersen (2014)). Indeed, BBMT show that beta pricing is more pronounced for stocks with smaller institutional holdings, consistent with our notion that $G$ traders (likely non-institutions) facilitate negative beta pricing.

We next show that under reasonable conditions, our analysis also accords with Ang, Hodrick, Xing, and Zhang (2006), who demonstrate a negative cross-sectional relation between idiosyncratic volatility and average returns. For the $j$'th basic security, if one runs a time series regression of $\tilde{R}_j$ against $\tilde{R}_M$, then the variance of the residual, which we refer to as the square idiosyncratic volatility (or simply IVOL), equals

$$IVOL_j = \text{Var}(\tilde{R}_j) - \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_M)^2}{\text{Var}(\tilde{R}_M)}.$$  \hspace{1cm} (15)

We let $\nu_{\theta_j}$ vary while holding other exogenous parameters constant. Intuitively, the return IVOL should be positively related with the cash flow volatility measured by $\nu_{\theta_j}$. It turns out in our model that this is true of “typical securities,” i.e., those with small $\text{Var}(\tilde{R}_j)/\text{Var}(\tilde{R}_M)$. This implies that their value variation is relatively low compared to the value variation of the market portfolio (in a sense formalized in Appendix A). The following lemma formalizes this observation:
**Lemma 2** Consider two typical basic securities, \( j \) and \( j' \), with \( \nu_{z_j} = \nu_{z_j'}, \bar{\xi}_j = \bar{\xi}_{j'} \), but \( \nu_{\theta_j} > \nu_{\theta_{j'}} \). Then, the idiosyncratic volatility \( \text{IVOL}_j > \text{IVOL}_{j'} \).

We then have the following proposition:

**Proposition 4** For \( G_j = G(\nu_{\theta_j}) \) given in Eq. (13), the \( \beta \)-adjusted expected return, 
\[-\gamma(\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j)) \bar{\xi}_j, \]
decreases in \( \nu_{\theta_j} \).

Lemma 2 and Proposition 4 imply that for typical basic securities, there is a negative relation (induced by \( \nu_{\theta_j} \)) between IVOL and the \( \beta \)-adjusted expected return. This is broadly consistent with Ang, Hodrick, Xing, and Zhang (2006), who show that stocks with high idiosyncratic volatility earn lower average returns. Proxying for total volatility by \( \nu_{\theta_j} \), our analysis also accords with Baker and Haugen (2012) who show that low risk stocks outperform high risk stocks in the vast majority of international equity markets.

The above analysis indicates that total volatility is negatively priced in the cross-section. However, in aggregate, risk is positively priced. To see this, note from Proposition 1 that the return of the market portfolio (over the risk free interest rate which is normalized to be zero) is given by

\[
\tilde{R}_M = \sum_{j=1}^{K+N} (\bar{\xi}_j + \tilde{z}_j) \tilde{R}_j = \sum_{j=1}^{K+N} [(\bar{\xi}_j + \tilde{z}_j)(\bar{\theta}_j + \gamma a_j(G_j)(\bar{\xi} + \tilde{z}_j))].
\]  

(16)

The following proposition can readily be derived.

**Proposition 5** The market risk premium, \( E(\tilde{R}_M) \), is positive.

Thus, our model is consistent with the negative pricing of volatility in the cross-section, but a positive pricing of risk in the aggregate (Haugen and Baker (2010), Ang, Hodrick, Xing, and Zhang (2006), and Mehra and Prescott (1985)).
2.7 Back to the Original Securities

The previous analysis focused on the basic securities for tractability. We now show that our main results carry over to the original securities. We can use Eq. (1) to reconstruct the original risky assets using the basic securities. Specifically, consider the $j$'th original risky asset as a portfolio of $\bar{V}_j$ units of the risk free asset, $\beta_{jk} \ (k = 1, \ldots, K)$ units of the $k$'th basic security, and one unit of the $j$'th basic security. Note from Proposition 1 that the return of the original risky asset can be expressed as:

$$\tilde{R}_j = \sum_{k=1}^{K} \beta_{jk} \left[ \tilde{\theta}_k + \gamma a_k(G_k)(\tilde{\xi}_k + \tilde{z}_k) \right] + \left[ \tilde{\theta}_j + \gamma a_j(G_j)(\tilde{\xi}_j + \tilde{z}_j) \right]. \tag{17}$$

The expected return of the $j$'th original risky asset is given by

$$E(\tilde{R}_j) = \sum_{k=1}^{K} \beta_{jk} \left[ \gamma a_k(G_k)\bar{\xi}_k \right] + \left[ \gamma a_j(G_j)\bar{\xi}_j \right]. \tag{18}$$

The expected return in our model takes a similar form as multi-factor models such as ICAPM (Merton (1973)) and APT (Ross (1976)). It follows from Corollary 1 that this expected return decreases in $G_k \ \forall k = 1, \ldots, K$ and $G_j$, and increases in $\rho$ (so long as $\beta_{jk} > 0 \ \forall k = 1, \ldots, K$).

The volatility of the $j$'th original risky asset is given by

$$\text{Var}(\tilde{R}_j) = \sum_{k=1}^{K} \beta_{jk}^2 \left[ \nu_{\theta_k} + (\gamma a_k(G_k))^2 \nu_{z_k} \right] + \left[ \nu_{\theta_j} + (\gamma a_j(G_j))^2 \nu_{z_j} \right].$$

It follows from Corollary 2 that this volatility decreases in $G_k \ \forall k = 1, \ldots, K$ and $G_j$, and increases in $\rho$.

Factor loadings can predict expected returns because the factor premium, $\gamma a_k(G_k)\bar{\xi}_k$, is identical across all assets. Can $\beta_{jM}$ (i.e., the beta with respect to the market return) also predict returns? One can estimate $\beta_{jM}$ by regressing $\tilde{R}_j$ against the returns on factor mimicking portfolio and the market portfolio. From Eq. (18), it also follows that after
adjusting for factor returns, the $j$’th original risky asset’s return is identical to the $j$’th basic security’s return. Thus, the market beta of a $j$’th original security is identical to that of the $j$’th basic security given in Eq. (7):

$$\beta_{jM} = \frac{\nu_{\theta_j} \bar{\xi}_j + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} \bar{\xi}_j}{\text{Var}(\tilde{R}_M)}.$$  \hspace{1cm} (19)

Plugging into Eq. (18) yields

$$E(\tilde{R}_j) = \sum_{k=1}^{K} \beta_{jk} [\gamma a_k(G_k) \bar{\xi}_k] + \frac{\gamma a_j(G_j) \text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}} \beta_{jM}.$$  

Proposition 2 indicates that $\lambda_j = \frac{\gamma a_j(G_j) \text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}}$, the slope of the relation between $E(\tilde{R}_j)$ and $\beta_{jM}$, decreases in $G_j$ and can be as low as zero.

One can estimate IVOL$_j$ by regressing $\tilde{R}_j$ on factor mimicking portfolios’ and the market portfolio’s returns. From Eq. (17) we see that after adjustment for the factor returns, this IVOL$_j$ is identical to the IVOL of the $j$’th basic security given in Eq. (15), which as argued in Lemma 2, increases in $\nu_{\theta_j}$. Use Eqs. (18) and (19) to write

$$E(\tilde{R}_j) = \sum_{k=1}^{K} \beta_{jk} [\gamma a_k(G_k) \bar{\xi}_k] + \gamma \text{Var}(\tilde{R}_M) \beta_{jM} - \gamma [\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j)] \bar{\xi}_j,$$

where the last term, $-\gamma [\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j)] \bar{\xi}_j$, represents the $\beta$-adjusted expected return. It follows from Proposition 4 that if $G_j = \mu \nu_{\theta_j}$ where $\mu < \gamma^2$ is a positive constant, then the $\beta$-adjusted expected return, $-\gamma [\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j)] \bar{\xi}_j$, decreases in $\nu_{\theta_j}$.

Thus, there is a negative relation (induced by $\nu_{\theta_j}$) between IVOL and the $\beta$-adjusted expected return. Note that our result in Proposition 5 on the market risk premium being positive continues to hold for the original securities. The reason is that the market portfolio of the original securities is just a reshuffle of the basic securities, and is therefore identical to the market portfolio of the basic securities.

All of the preceding analysis indicates that results for the mimicking portfolios generally carry over to the original securities. Our work suggests the following untested
empirical implications, which rely on the premise that stocks with more retail traders are likely to also have more $G$ traders. We predict that *ceteris paribus*, stocks with proportionally more retail traders are less volatile and more actively traded. The cross-sectional (negative) relation between volatility and future returns is also likely to be more evident in stocks actively traded and held by retail investors.

3 Comparing to the Economy With No or Partial Presence of $G$ Traders

We now compare the equilibria with (i) complete absence of the $G$ traders and (ii) presence of the $G$ traders in some, but not all, securities. For simplicity, the analysis in this section is focused on the basic securities.

3.1 Comparing to the Economy With No $G$ Traders

Consider two economies. In the first economy, all agents are non-$G$ traders, while in the second, all are $G$ traders. For a variable $\eta$ in the basic economy, we use $\eta^{A,G}$ and $\eta^{A,NG}$ to indicate its counterpart in the all-$G$ and all-non-$G$ economies.

The first economy, the all-non-$G$ economy, is equivalent to the basic economy with $\rho = 1$ and $a_{j}(G_{j}) = \nu_{0_{j}}$. Using a similar derivation as that for Proposition 1, we can show that the price and return of the $j$’th basic security are given by

\[
P_{j}^{A,NG} = -\gamma\nu_{0_{j}}(\bar{\xi}_{j} + \bar{z}_{j}),
\]
\[
\tilde{R}_{j}^{A,NG} = \tilde{\theta}_{j} + \gamma\nu_{0_{j}}(\bar{\xi}_{j} + \bar{z}_{j}).
\]  

(20)

The return of the market portfolio is $\tilde{R}_{M}^{A,NG} = \sum_{j=1}^{K+N} (\bar{\xi}_{j} + \bar{z}_{j})\tilde{R}_{j}^{A,NG}$.

Similar to Eq. (7), the covariance between the returns of the basic security and the
market portfolio is given by

\[
\text{Cov}(\tilde{R}_j^{A,NG}, \tilde{R}^{A,NG}_M) = \text{Cov}(\tilde{R}_j^{A,NG}, (\bar{\xi} + \tilde{z})\tilde{R}^{A,NG}_j) = \nu_{\theta_j}\bar{\xi}_j + 2\gamma^2\nu_{\theta_j}^2\nu_{z_j}\bar{\xi}_j.
\]

Then, the expected return of the basic security can be expressed as:

\[
E(\tilde{R}_j^{A,NG}) = \gamma\nu_{\theta_j}\bar{\xi}_j = \frac{\gamma\nu_{\theta_j}\text{Var}(\tilde{R}^{A,NG}_M)}{\nu_{\theta_j} + 2\gamma^2\nu_{\theta_j}^2\nu_{z_j}}\beta_{jM}^{A,NG},
\]

where \(\beta_{jM}^{A,NG} = \frac{\text{Cov}(\tilde{R}_j^{A,NG}, \tilde{R}^{A,NG}_M)}{\text{Var}(\tilde{R}^{A,NG}_M)}\).

Let \(\lambda_{jM}^{A,NG} = \frac{\gamma\nu_{\theta_j}\text{Var}(\tilde{R}^{A,NG}_M)}{\nu_{\theta_j} + 2\gamma^2\nu_{\theta_j}^2\nu_{z_j}}\) denote the slope of the relation between \(E(\tilde{R}_j^{A,NG})\) and \(\beta_{jM}^{A,NG}\). An obvious observation is that \(\lambda_{jM}^{A,NG} > 0\). Therefore, \(\beta\)'s still have power to predict stock returns. If \(\nu_{z_j} = 0\), then \(\lambda_{jM}^{A,NG} = \gamma\text{Var}(\tilde{R}^{A,NG}_M)\) is identical across all assets. In this case, \(\beta\)'s are the only predictive variable for expected returns.

The second economy, the all-\(G\) economy, is equivalent to the basic economy with \(\rho = 0\) and \(a_j(G_j) = \nu_{\theta_j} - G_j/\gamma^2\). Using a similar derivation as that for Proposition 1, we can show that the price and return of the \(j\)'th basic security is given by

\[
\begin{align*}
P_j^{A,G} &= -\gamma(\nu_{\theta_j} - G_j/\gamma^2)(\bar{\xi}_j + \tilde{z}_j),
\tilde{R}_j^{A,G} &= \bar{\theta}_j + \gamma(\nu_{\theta_j} - G_j/\gamma^2)(\bar{\xi}_j + \tilde{z}_j).
\end{align*}
\]

(21)

The return of the market portfolio is \(\tilde{R}^{A,G}_M = \sum_{j=1}^{K+N} (\bar{\xi}_j + \tilde{z}_j)\tilde{R}_j^{A,G}\).

Similar to Eq. (7), the covariance between the returns of the basic security and the market portfolio is given by

\[
\text{Cov}(\tilde{R}_j^{A,G}, \tilde{R}^{A,G}_M) = \text{Cov}(\tilde{R}_j^{A,G}, (\bar{\xi} + \tilde{z})\tilde{R}^{A,G}_j) = \nu_{\theta_j}\bar{\xi}_j + 2\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j}\bar{\xi}_j.
\]

Then, the expected return of the basic security can be expressed as:

\[
E(\tilde{R}_j^{A,G}) = \gamma(\nu_{\theta_j} - G_j/\gamma^2)\bar{\xi}_j = \frac{\gamma(\nu_{\theta_j} - G_j/\gamma^2)\text{Var}(\tilde{R}^{A,G}_M)}{\nu_{\theta_j} + 2\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j}}\beta_{jM}^{A,G},
\]

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where \( \beta_{jM}^{AG} = \frac{\text{Cov}(\tilde{R}_{j}^{AG}, \tilde{R}_{M}^{AG})}{\text{Var}(\tilde{R}_{M}^{AG})} \).

We compare the two economies in the following proposition.

**Proposition 6**

(i) \( E(\tilde{R}_{j}^{A,NG}) > E(\tilde{R}_{j}^{AG}) \) and \( \text{Var}(\tilde{R}_{j}^{A,NG}) > \text{Var}(\tilde{R}_{j}^{AG}) \). Thus, the \( j \)’th basic security has higher expected return and volatility in the all-non-\( G \) economy than in the all-\( G \) economy.

(ii) \( E(\tilde{R}_{M}^{A,NG}) > E(\tilde{R}_{M}^{AG}) \) and \( \text{Var}(\tilde{R}_{M}^{A,NG}) > \text{Var}(\tilde{R}_{M}^{AG}) \). Thus, the market portfolio has higher expected return and volatility in the all-non-\( G \) economy than in the all-\( G \) economy.

(iii) \( \lambda_{j}^{A,NG} > \lambda_{j}^{AG} \). Thus, \( \beta \)’s have more predictive power in the all-non-\( G \) economy than in the all-\( G \) economy.

In general, within the all-\( G \) economy, risk premiums (and volatilities) are attenuated because of the \( G \)-traders’ penchant to buy low and sell high, which, in turn, attenuates the pricing of risk. This suggests the testable implication that economies which are dominated by retail investors should exhibit lower volatility and less evidence of covariance risk pricing. Indeed, under the assumption that retail investors are more likely to participate during periods of positive sentiment (Grinblatt and Keloharju (2001) and Lamont and Thaler (2003)), and less likely to participate during periods of negative sentiment, our analysis accords with Antoniou, Doukas, and Subrahmanyam (2016) who show that covariance risk pricing is more prevalent during periods of low sentiment and vice versa.
3.2 Comparing to the Economy With Partial Presence of G Traders

Consider a hybrid case in which G traders are present in the trading of the basic securities mimicking the $K$ factors and the first $N_1$ residuals. They are not present in the remaining $N - N_1$ basic securities. Like the basic economy, there are a mass $\rho$ of non-G traders and a mass $1 - \rho$ of G traders.

We continue to write the return of the $j$’th basic security as $\tilde{R}_j = \tilde{\theta}_j - P_j$. (For convenience, we use the same notation for prices in all securities, i.e., $P_j$ for the $j$’th security, even though prices of securities without G traders can take a different form from that in Subsection 2.4.) The market portfolio has a return $\tilde{R}_M = \sum_{j=1}^{K+N} (\bar{\xi}_j + \tilde{z}_j)\tilde{R}_j$.

For the first $K+N_1$ basic securities, i.e., $\forall j \leq K+N_1$, our analysis in the subsection 2.4 still holds. Particularly, the expected return of the $j$’th basic security takes the form

$$E(\tilde{R}_j) = \lambda_j \beta_{jM}.$$

where $\lambda_j = \frac{\gamma a_j(G_j)\text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}}$ denote the slope of the relation between $E(\tilde{R}_j)$ and $\beta_{jM}$.

For the remaining $N - N_1$ basic securities, i.e., $\forall j > K+N_1$, there is only a mass $\rho$ of non-G traders to clear the market. Using a similar analysis as in Subsection 2.4, we can show that for these securities,

$$P_j^o = -(\gamma/\rho)\nu_{\theta_j}(\bar{\xi}_j + \tilde{z}_j),$$

$$\tilde{R}_j^o = \tilde{\theta}_j + (\gamma/\rho)\nu_{\theta_j}(\bar{\xi}_j + \tilde{z}_j).$$

[Here, we use the superscript-$^o$ to indicate the $N - N_1$ securities.] Similar to Eq. (7), the covariance between the returns of the basic security and the market portfolio is given by

$$\text{Cov}(\tilde{R}_j^o, \tilde{R}_M) = \nu_{\theta_j} \bar{\xi}_j + 2(\gamma/\rho)^2 \nu_{\theta_j}^2 \nu_{z_j} \bar{\xi}_j.$$
Then, the expected return of the basic security can be expressed as:

\[ E(\tilde{R}_j) = \frac{(\gamma/\rho)\nu_0 \lambda_j \bar{\rho}}{\nu_j + 2(\gamma/\rho)^2 \nu^2_0 \nu_j} \beta_{jM}^o = \lambda_j^o \beta_{jM}^o, \]

where \( \beta_{jM}^o = \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} \), and \( \lambda_j^o = \frac{(\gamma/\rho)\nu_0 \lambda_j \text{Var}(\tilde{R}_M)}{\nu_j + 2(\gamma/\rho)^2 \nu^2_0 \nu_j} \) denotes the slope of the relation between \( E(\tilde{R}_j) \) and \( \beta_{jM}^o \). We then have the following proposition.

**Proposition 7** Consider two basic securities, \( j \) and \( j' \), with \( \nu_j = \nu_{j'}, \nu_{z_j} = \nu_{z_{j'}}, \bar{\xi}_j = \bar{\xi}_{j'} \), but the \( j \)’th (\( j' \)’th) security is (is not) traded by \( G \) traders. Then, \( \lambda_j < \lambda_{j'}^o \). Thus, the presence of \( G \) traders reduces the predictive power of \( \beta \)’s.

The above proposition implies that beta pricing will be less evident in securities that are traded relatively more by \( G \) traders. Again, the notion is simply that \( G \) traders, via their more aggressive trading in securities where they are present, attenuate the pricing of risk. The above proposition indicates cross-sectional variation in risk pricing according to whether \( G \) traders are more or less likely to be present. Thus, if retail investors are more likely to be present in visible, brand name stocks (Frieder and Subrahmanyam (2005)), then covariance risk pricing will be less evident in these stocks.

## 4 Stock Market Versus Other Gambling Avenues

Agents in the real world have a large number of opportunities to indulge in risky gambles; for example, gambling in casinos or participating in lotteries. An interesting issue is the incentives to substitute across various ways to gamble in risky opportunities. Accordingly, in this section, we introduce a new class of agents, a mass \( \rho^A \) of \( G^A \) traders. We assume that a \( G^A \) trader is endowed with \( M_i \) units of the risk free asset. If a \( G^A \) trader invests in the stock market, then he has the same utility function as \( G \) traders, that is, considering
the basic securities, we have that

\[ U_{G^A}(W_{i1}, X_i) = -\exp\left[-\gamma W_{i1} - 0.5 \sum_{j=1}^{N+K} (G_j X_{ij})^2\right], \]

where \( W_{i1} = \bar{M}_i + \sum_{j=1}^{N+K} X_{ij}(\bar{\theta}_j - P_j) \), and \( X_{ij} \) is his demand for the \( j \)'th basic security.

Note that \( \bar{\theta}_j \sim \mathcal{N}(0, \nu_{\theta_j}) \). It follows, after taking expectations over \( \bar{\theta}_j \), that

\[ E\left[U_{G^A}(W_{i1}, X_i)\right] = -\exp\left[-\gamma \bar{M}_i - \gamma \sum_{j=1}^{N+K} X_{ij}(-P_j) + 0.5 \gamma^2 \sum_{j=1}^{N+K} (\nu_{\theta_j} - G_j/\gamma^2) X_{ij}^2\right]. \] (22)

Let \( \rho_1 \in [0, \rho^A] \) (which is to be determined) be the mass of \( G^A \) traders who choose to invest in the stock market. It follows a similar analysis as in the subsection 2.2 that each non-\( G \), \( G \), and \( G^A \) trader’s demands for the \( j \)'th basic security are respectively given by

\[ X_{NG,j}(P_j) = \frac{-P_j}{\gamma \nu_{\theta_j}}, \] (23)

\[ X_{G,j}(P_j) = \frac{-P_j}{\gamma(\nu_{\theta_j} - G_j/\gamma^2)}, \] (24)

\[ X_{G^A,j}(P_j) = \frac{-P_j}{\gamma(\nu_{\theta_j} - G_j/\gamma^2)}. \] (25)

The market clearing condition, \( \bar{\xi}_j + \bar{z}_j = \rho X_{NG,j}(P_j) + (1 - \rho) X_{G,j}(P_j) + \rho_1 X_{G^A,j}(P_j) \), implies that the price of the \( j \)'th basic security is given by

\[ P_j = -\gamma a_j(G_j, \rho_1)(\bar{\xi}_j + \bar{z}_j), \] (26)

where \( a_j(G_j, \rho_1) = \frac{1}{\rho \nu_{\theta_j} + \frac{1 - \rho + \rho_1}{\nu_{\theta_j} - G_j/\gamma^2}} \). Substituting for the \( G^A \) trader’s optimal demand, \( X_{G^A,j}(P_j) \) in Eq. (25), where \( P_j \) is given in Eq. (26), into his expected utility function (Eq. 22), we obtain

\[ E\left[U_{G^A}(W_{i1}, X_i)\right] = -\exp\left(-\gamma \bar{M}_i - 0.5 \sum_{j=1}^{K+N} \frac{P_j^2}{\nu_{\theta_j} - G_j/\gamma^2}\right) \]

\[ = -\exp\left[-\gamma \bar{M}_i - 0.5 \sum_{j=1}^{K+N} \gamma^2 a_j(G_j, \rho_1)^2(\bar{\xi}_j + \bar{z}_j)^2\right]. \] (27)
$G^A$ traders may choose to gamble through an alternative venue, such as a casino or a statewide lottery, instead of investing in the stock market. If a $G^A$ trader chooses to gamble through this venue, then his utility function takes the following form:

$$U_{G^A}(W_{i1}, Y) = -\exp(-\gamma W_{i1} - 0.5Y^2 G^A),$$

where $W_{i1} = \bar{M}_i + Y\bar{\mu}$. $\bar{\mu}$ represents the payoff of this gambling opportunity (we normalize its mean to be zero). $Y > 0$ is a positive constant representing his fixed position. $G^A$ represents the parameter that governs his utility derived from this gambling opportunity. It follows, after taking expectations over $\bar{\mu}$, that

$$E\left[U_{G^A}(W_{i1}, Y)\right] = -\exp\left[ -\gamma \bar{M}_i + 0.5\gamma^2 Y^2 (\nu_\mu - G^A/\gamma^2) \right]. \quad (28)$$

We assume that a $G^A$ trader can either invest in the stock market or gamble through the alternative venue (but not both) based on the comparison between the ex ante expected utility from trading the stocks (taking expectations over $\bar{z}_j$ in Eq. 27) and the expected utility from the alternative venue in Eq. (28). One then obtains the following proposition:

**Proposition 8** The mass $\rho_1$ of $G^A$ traders who trade stocks decreases in $G^A$.

If the alternative gambling opportunity becomes more attractive (higher $G^A$), then less $G^A$ traders will trade stocks.

Using Eqs. (23)-(25), we can express the total expected trading volume in the basic security as

$$T_j(\rho_1) \equiv 0.5\rho E\left[|X_{NG,j}(P_j) - (\bar{\xi}_j + \bar{z}_j)|\right] + 0.5(1 - \rho)E\left[|X_{G,j}(P_j) - (\bar{\xi}_j + \bar{z}_j)|\right] + 0.5\rho_1 E\left[|X_{G^A,j}(P_j)|\right]. \quad (29)$$

We then have the following result.
Corollary 4  The expected trading volume, $T_j(\rho_1)$, increases in $\rho_1$, and decreases in $G^A$.

We thus find that substituting across various avenues can affect trading volume in equities. Specifically, increased attractiveness of the alternative gambling opportunity causes a decrease in equilibrium equity volume. This result is consistent with Gao and Lin (2015). They show that equity trading volume in Taiwan decreases as the total jackpot of a major statewide lottery increases.

5  A Dynamic Extension: Equilibrium Where Trading Value Depends on Past Market Outcomes

We now consider a dynamic extension of our setting where the utility from trading depends on past profits. Specifically, we model the notion that if an agent earns positive profits, he may derive greater utility from gambling in the stock market (see, for example, Thaler and Johnson (1990) or Coventry and Constable (1999)). We show that the “excitement” created by positive profits can lead to an overreaction to mildly positive information and thus cause a “bubble” in stock prices. Our rationale for this bubble is consistent with experimental arguments that emotional excitement can cause bubbles (Bellotti, Taffler, and Tian (2010) and Andrade, Odean, and Lin (2015)); in our model, the pathway is that positive profits increase the “excitement” or utility from additional trading and thus cause an overreaction to mildly positive information.

We assume that a risky security is traded at Dates $t = 0, 1, 2, \text{and } 3$ and revert to the case of symmetric information. While we analyze a single security, without loss of generality, it can be viewed as any “basic security” as defined in Section 2.2. The supply of the risky security is a positive constant, $\xi$.\textsuperscript{13} For convenience, as in the previous section,\textsuperscript{13}

\textsuperscript{13}We have verified that generalizing to asymmetric information and random supply is possible, but we lose closed-form solutions, without the promise of additional intuition.
we suppress $j$ subscripts. At Date 3, the security pays off a liquidation dividend

$$V = \bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3.$$ 

$\bar{V}$ is a positive constant, which represents the expected dividend. The variables $\tilde{\theta}_t$, $t = 1, 2, 3$, represent exogenous cash flow shocks, which are mutually independent and multivariate normally distributed with mean zero. $\tilde{\theta}_t$’s are public signals released at Dates $t = 1, 2$. Its prices are $P_t$ at Dates $t = 0, 1, 2$.

There is a mass unity of identical agents who trade the risky security. At Date 0, they all hold an identical long position, in aggregate $\bar{\xi}$, to clear the market. At Date 1, if $P_1 > P_0$, they make money at Date 1; otherwise, they do not. If $P_1 > P_0$, so that they make money at Date 1, then after Date 1 and before Date 2, with probability $\rho$, an agent becomes a non-\textit{G} trader; with probability $1 - \rho$, he becomes a \textit{G} trader. If $P_1 \leq P_0$ and the agents do not make money at date 1, then all agents remain non-\textit{G} traders with a probability of unity.

The $i$’th non-\textit{G} trader’s utility function is the standard exponential:

$$U(W_{i3}) = -\exp(-\gamma W_{i3}),$$

where $W_{i3}$ is his final wealth, and $\gamma$ is a positive constant representing the absolute risk aversion coefficient. The $i$’th \textit{G} trader’s utility function takes the form

$$U_{G}(W_{i3}, X_{i2}) = -\exp(-\gamma W_{i3})\exp(-0.5GX_{i2}^2),$$

where $X_{i2}$ is the quantity of risky security he has bought at date 2 and continues to hold until the end of the game, and $G$ is a positive constant.

The $i$’th trader is endowed with $\bar{X}_{i0}$ units of risky securities. For convenience, we let $\tilde{\theta}_t$’s, $t = 1, 2, 3$, have the same variance $\nu_{\theta}$. Let the price and return of the risk free asset be 1. We then have the following result:
Proposition 9 There is an equilibrium characterized by the following prices:

- $P_0$ is given by

\[
P_0 = \bar{V} + H_\theta - 2\gamma \nu_\theta \bar{\xi},
\]

where $a(G) = \frac{1}{\nu_\theta + \frac{1-\rho}{\nu_\theta - G/\gamma^2}}$. The variable $H_\theta$ is uniquely determined by

\[
0 = \int_{H_\theta}^{\infty} \frac{\bar{\theta}_1 - H_\theta + \gamma(\nu_\theta - a(G))\bar{\xi}}{\exp(\gamma \bar{\xi} \bar{\theta}_1)} d\Phi(\bar{\theta}_1)
\]

\[
+ \int_{-\infty}^{H_\theta} \frac{\bar{\theta}_1 - H_\theta}{\exp(\gamma(\bar{\theta}_1 - \gamma(\nu_\theta - a(G))\bar{\xi}))} \exp(-0.5 \frac{(\gamma \nu_\theta \bar{\xi})^2}{\nu_\theta}) d\Phi(\bar{\theta}_1 / \sqrt{\nu_\theta}).
\]

and $\Phi(.)$ is the cumulative density function of standard normal distribution.

- If $\bar{\theta}_1 > H_\theta$, then

\[
P_1 = \bar{V} + \bar{\theta}_1 - \gamma \nu_\theta \bar{\xi} - \gamma a(G)\bar{\xi},
\]

\[
P_2 = \bar{V} + \bar{\theta}_1 + \bar{\theta}_2 - \gamma a(G)\bar{\xi}.
\]

Because $P_1 > P_0$, a mass $1 - \rho$ of traders convert to $G$ traders immediately following Date 1.

- If $\bar{\theta}_1 \leq H_\theta$, then

\[
P_1 = \bar{V} + \bar{\theta}_1 - 2\gamma \nu_\theta \bar{\xi},
\]

\[
P_2 = \bar{V} + \bar{\theta}_1 + \bar{\theta}_2 - \gamma \nu_\theta \bar{\xi}.
\]

Because $P_1 \leq P_0$, all traders remain non-$G$ traders throughout the timeline.
Here is a sketch of the proof of this proposition (the formal proof is in the Appendix). We use backward induction. There are three steps. In the first step, we study the equilibrium demands and prices at Dates 1 and 2 conditional on the event that at Date 1, traders make money because $\tilde{\theta}_1 > H_\theta$ so $P_1 > P_0$ (call this Regime 1). Note that in this regime, at Date 2, there is a mass $\rho (1 - \rho)$ of non-$G$ traders ($G$ traders). At Date 1, an agent knows that he will be a non-$G$ trader ($G$ trader) with probability $\rho (1 - \rho)$. In the second step, we study the equilibrium demands and prices at dates 1 and 2 conditional on the event that at Date 1, traders do not make money because $\tilde{\theta}_1 \leq H_\theta$ so $P_1 \leq P_0$ (call this Regime 2). This step is simpler than the first step because all traders are non-$G$ traders. In the third step, we focus on Date 0, and derive the expressions for $P_0$ and the threshold $H_\theta$.

There are two interesting results. The first relates to the price reaction for $\tilde{\theta}_1$ around the threshold $H_\theta$:

\[
P_1(\tilde{\theta}_1 \searrow H_\theta) = \bar{V} + \tilde{\theta}_1 - \gamma \nu_\theta \bar{\xi} - \gamma a(G) \bar{\xi},
\]

\[
P_1(\tilde{\theta}_1 \nearrow H_\theta) = \bar{V} + \tilde{\theta}_1 - 2\gamma \nu_\theta \bar{\xi}.
\]

It is easy to show that

\[
P_1(\tilde{\theta}_1 \searrow H_\theta) - P_1(\tilde{\theta}_1 \nearrow H_\theta) = \gamma \nu_\theta \bar{\xi} - \gamma a(G) \bar{\xi} > 0,
\]

because $a(G) < a(0) = \nu_\theta$. This suggests a small $\tilde{\theta}_1$ (e.g., earnings) can induce a significant price movement. Another interpretation of this observation is that a relatively minor piece of news can cause substantial moves in prices.

The second result relates to long-run performance. If $\tilde{\theta}_1 > H_\theta$, then the subsequent returns are

\[
P_2 - P_1 = \tilde{\theta}_2 + \gamma \nu_\theta \bar{\xi}, \text{ and } V - P_2 = \tilde{\theta}_3 + \gamma a(G) \bar{\xi}.
\]
If $\tilde{\theta}_1 \leq H_\theta$, then the subsequent returns are

$$P_2 - P_1 = \tilde{\theta}_2 + \gamma \nu_\theta \xi,$$

and

$$V - P_2 = \tilde{\theta}_3 + \gamma \nu_\theta \xi.$$ 

A comparison between these two cases suggests that if $\tilde{\theta}_1 > H_\theta$, then there is a long-run underperformance because $a(G) < a(0) = \nu_\theta$. Thus, a minor piece of good news can cause securities to become dramatically overpriced and thus exhibit subpar returns in the long run.

Figure 1 plots the price paths conditional on the public announcement $\tilde{\theta}_1$. We assume the parameter values $\bar{V} = 5$, $\nu_\theta = 1$, $\xi = 1$, $\gamma = 0.5$, $\rho = 0.5$, and $G = 0.2$. The realizations of $\tilde{\theta}_2$ and $\tilde{\theta}_3$ are assumed to be zero, i.e., their mean. This implies that the threshold $H_\theta = -0.388$. Moving from the bottom to the top, each path in the figure represents a realization of $\tilde{\theta}_1$ from -1 to 0 (step size=0.025). $\tilde{\theta}_1 \leq H_\theta$ for the paths indicated by $\triangle$’s. $\tilde{\theta}_1 > H_\theta$ for the paths indicated by *’s. We see that if $\tilde{\theta}_1$ is below the threshold $H_\theta = -0.388$, the price reaction to $\tilde{\theta}_1$ is non-positive. Once $\tilde{\theta}_1$ has surpassed the threshold $H_\theta = -0.388$, the price reaction becomes positive.

Particularly, look at the two paths bordering the hollow area. The south path is for $\tilde{\theta} = -0.4$. The north path is for $\theta = -0.375$. Although $\tilde{\theta}_1$ differs by only 0.025 across the two path groups, the price reactions are very different. On both paths, $P_0 = 3.612$. However, on the south path, $P_1 = 3.6$ so $P_1 - P_0 = -0.012$; on the north path, $P_1 = 3.9583$ so $P_1 - P_0 = 0.3463$. The difference in the price reaction, $P_1 - P_0$, equals 0.3583, which is more than fourteen times the difference in $\tilde{\theta}_1$ (0.025).

The immediate return subsequent to the release of $\tilde{\theta}_1$, $P_2 - P_1 = 0.5$, is identical across all $\tilde{\theta}_1$ paths. But the long-run performance for the paths with $\tilde{\theta}_1 > H_\theta$ indicated by *’s, $V - P_2 = 0.1667$, is lower than that for the paths with $\tilde{\theta}_1 \leq H_\theta$ indicated by $\triangle$’s, $V - P_2 = 0.5$. This indicates long-run underperformance following a good public announcement. The underperformance is characteristic of bubble-like episodes in the
stock market (such as the technology bubble of the 1990s, viz. Brunnermeier and Nagel (2004)), whereas the positive event (that creates the bubble) could be something as simple as good initial sales or earnings figures for the relevant sector.

More generally, the preceding analysis suggests a testable implication. Specifically, for stocks that are popular amongst retail investors, we predict a nonlinear response to positive news, that is a small reaction to modest news announcement, but a disproportionately larger reaction to major (positive) announcements. Following the large positive announcements, these stocks should exhibit long-run reversals (conditional on the news).

6 Conclusion

In this paper, we present a model where agents derive direct utility from trading. We show that the presence of such agents causes assets to be overpriced, attenuates beta pricing and volatility, and raises trading volume in financial markets. Assets with high trading volume earn lower expected returns. Assuming that agents derive greater utility from trading more volatile stocks, our model accords with a set of intriguing empirical findings: Volatility is priced negatively in the cross-section, but positively in the aggregate (viz. Haugen and Baker (2010)). Further, the presence of agents who receive direct utility from trading causes “bubbles,” i.e., overreactions in asset prices if agents’ utility from trading depends on past profit outcomes in financial markets. We also show that the relative attractiveness of alternative opportunities to “gamble” (indulge in risky activities for pleasure) affects trading volume in equities.

Untested implications of our analysis are that stocks that are popular amongst retail investors should exhibit weaker evidence of covariance risk pricing and lower volatility, with greater trading activity. These stocks should also exhibit disproportionate price reactions to moderately positive news announcements. The analysis, under reasonable
additional assumptions, also accords with a variety of documented stylized facts: the negative relation between average returns and volume as well as idiosyncratic (or total) volatility (Datar, Naik, and Radcliffe (1998), Ang, Hodrick, Xing, and Zhang (2006), Baker and Haugen (2012)), the lack of evidence consistent with covariance risk pricing (Fama and French (1992)), the pricing of covariance risk conditional on sentiment (Antoniou, Doukas, and Subrahmanyam (2016)), and the rise of volatility in conjunction with the rise in institutional holdings (Campbell, Lettau, Malkiel, and Xu (2001) and Malkiel and Xu (1999)).

Our work raises many issues. First, it would be interesting to examine a fully dynamic model with exits and entry by such agents. Second, it may be interesting to combine trading for entertainment and other investor biases, such as representativeness and overconfidence, and to examine the market equilibrium that results. Finally, the specific factors that influence how much utility is derived per unit trade (such as age, personality attributes) need to be considered in more depth within a theoretical setting. These and other issues are left for future research.
Appendix A

Proof of Corollary 1: From Proposition 1, \( a_j(G_j) = \frac{1}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2} \). It is easy to show after taking derivatives that \( a_j(G_j) \) decreases in \( G_j \), and increases in \( \rho \). Finally, \( a_j(0) = \frac{1}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j}} = \nu_{\theta_j} \).

Q.E.D.

Proof of Proposition 2: (i) From Corollary 1, \( a_j(G_j) < a_j(G_j') \) because \( G_j > G_j' \). Note that \( \lambda_j = \frac{\gamma a_j(G_j) \text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}} \). For \( \lambda_j < \lambda_{j'} \), it suffices that

\[
\frac{a_j(G_j)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}} - \frac{a_j(G_j')}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j')^2 \nu_{z_j}}
\]

\[
\propto a_j(G_j) \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} \right] - a_j(G_j') \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j')^2 \nu_{z_j} \right]
\]

\[
= \left[ a_j(G_j) - a_j(G_j') \right] \left[ \nu_{\theta_j} - 2\gamma^2 a_j(G_j') a_j(G_j) \nu_{z_j} \right]
\]

\[
\propto 2\gamma^2 a_j(G_j') a_j(G_j) \nu_{z_j} - \nu_{\theta_j}
\]

\[
< 2\gamma^2 \nu_{\theta_j}^2 \nu_{z_j} - \nu_{\theta_j}
\]

\[
< 0,
\]

where the second “\( \propto \)” follows from \( a_j(G_j) < a_j(G_j') \), the first inequality follows from \( a_j(G_j), a_j(G_j') < \nu_{\theta_j} \) (see Corollary 1), and the last inequality obtains under the assumption \( \nu_{z_j} < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2) \) (see Condition (2)).

If \( G_j/\gamma^2 \not< \nu_{\theta_j} \), then \( a_j(G_j) = \frac{1}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2} \ll 0 \) so that \( \lambda_j \ll 0 \).

(ii) Consider the above expression for \( \lambda_j \). It follows after taking partial derivatives that \( \lambda_j \) increases in \( \text{Var}(\tilde{R}_M) \). \( \lambda_j \) also increases in \( a_j(G_j) \) because

\[
\frac{\partial \lambda_j}{\partial a_j(G_j)} \propto \nu_{\theta_j} - 2\gamma^2 a_j(G_j)^2 \nu_{z_j} > \nu_{\theta_j} - 2\gamma^2 \nu_{\theta_j}^2 \nu_{z_j} > 0,
\]

\[36\]
where the first inequality follows from $a_j(G_j) < \nu_{\theta_j}$ (see Corollary 1), and the last inequality obtains under the assumption $\nu_{z_j} < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2)$ (see Condition (2)).

Note that $\text{Var}(\tilde{R}_M)$ and $a_j(G_j)$ increase in $\rho$ (see Corollaries 1 and 2). It follows that $\lambda_j$ increases in $\rho$.

Q.E.D.

**Proof of Proposition 3:** (i) From Corollary 1, $a_j(G_j) < \nu_{\theta_j}$. Therefore, the $\beta$-adjusted expected return of the $j$'th basic security $-\gamma \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j) \right] \bar{\xi}_j < 0$.

(ii) From Corollary 1, $a_j(G_j) < a_j(G_{j'})$ because $G_j > G_{j'}$. The difference between the $\beta$-adjusted expected returns of the $j$'th and $j'$'th basic securities is

$$ -\gamma \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j) \right] \bar{\xi}_j - \gamma \left[ \nu_{\theta_{j'}} + 2\gamma^2 a_{j'}(G_{j'})^2 \nu_{z_j} - a_{j'}(G_{j'}) \right] \bar{\xi}_{j'} $$

$$ = \gamma \left[ a_j(G_j) - a_{j'}(G_{j'}) - 2\gamma^2(a_j(G_j)^2 - a_{j'}(G_{j'})^2) \nu_{z_j} - a_j(G_j) \right] \bar{\xi}_j $$

$$ \propto 2\gamma^2(a_j(G_j) - a_{j'}(G_{j'})) \nu_{z_j} - 1 $$

$$ < 4\gamma^2 \nu_{\theta_j} \nu_{z_j} - 1 $$

$$ < 0,$$

where the “$\propto$” follows from $a_j(G_j) < a_{j'}(G_{j'})$, the first inequality follows from $a_j(G_j), a_{j'}(G_{j'}) < \nu_{\theta_j}$ (see Corollary 1), and the last inequality obtains under the assumption $\nu_{z_j} < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2)$ (see Condition (2)).

(iii) It follows from the above proof for part (ii) that the $\beta$-adjusted expected return of the $j$'th basic security, $-\gamma \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j} - a_j(G_j) \right] \bar{\xi}_j$, increases in $a_j(G_j)$. Also, $a_j(G_j)$ increases in $\rho$ (see Corollary 1). Therefore, this $\beta$-adjusted expected return increases in $\rho$.

Q.E.D.
Proof of Corollary 3: Write Eqs. (10) and (11) as

\[
\begin{align*}
X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) & \sim N(A_{NG}\bar{\xi}_j, A_{NG}^2\nu_z), \\
X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) & \sim N(A_G\bar{\xi}_j, A_G^2\nu_z),
\end{align*}
\]

where

\[
A_{NG} \equiv \frac{a_j(G_j)}{\nu_{\theta_j}} - 1, \quad \text{and} \quad A_G \equiv \frac{a_j(G_j)}{\nu_{\theta_j} - G_j/\gamma^2} - 1.
\]

Here are some intermediate results we will use in the proof of this corollary. First, \(A_{NG} < 0\) and decreases in \(G_j\) from Corollary 1. Second, \(A_G = \frac{a_j(G_j)}{\nu_{\theta_j} - G_j/\gamma^2} - 1 = \frac{1}{\rho} \left( \frac{1}{\nu_{\theta_j} - G_j/\gamma^2} + 1 - \rho \right) > 0\), and increases in \(G_j\).

Using the fact listed in Footnote 14 and the above new expressions for the distributions of \(X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)\) and \(X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)\), we can express the total expected trading volume in the \(j\)'th basic security (Eq. (29)) as

\[
T_j = 0.5\rho E\left[ |X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)| \right] + 0.5(1 - \rho) E\left[ |X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)| \right]
\]

\[
= 0.5\rho (-A_{NG}\sqrt{\nu_z}) \left[ 2\phi(-\frac{\tilde{\xi}_j}{\sqrt{\nu_z}}) - \frac{\tilde{\xi}_j}{\sqrt{\nu_z}} (1 - 2\Phi(-\frac{\bar{\xi}_j}{\sqrt{\nu_z}})) \right]
\]

\[
+ 0.5(1 - \rho) A_G\sqrt{\nu_z} \left[ 2\phi(-\frac{\tilde{\xi}_j}{\sqrt{\nu_z}}) + \frac{\tilde{\xi}_j}{\sqrt{\nu_z}} (1 - 2\Phi(-\frac{\bar{\xi}_j}{\sqrt{\nu_z}})) \right]
\]

\[
= \left[ -\rho A_{NG} + (1 - \rho) A_G \right] \times 0.5\sqrt{\nu_z} \left[ 2\phi(-\frac{\tilde{\xi}_j}{\sqrt{\nu_z}}) - \frac{\tilde{\xi}_j}{\sqrt{\nu_z}} (1 - 2\Phi(-\frac{\bar{\xi}_j}{\sqrt{\nu_z}})) \right].
\]

Footnote 14 indicates that the values in the brackets are positive. From the above analysis, \(A_{NG}\) decreases in \(G_j\), and \(A_G\) increases in \(G_j\). Therefore, \(T_j\) increases in \(G_j\).

Q.E.D.

---

\textsuperscript{14} If \(y \sim N(\bar{y}, \nu)\), then \(E[|y|] = \sqrt{\nu} \left[ 2\phi\left( \frac{\bar{y}}{\sqrt{\nu}} \right) + \bar{y} (1 - 2\Phi\left( \frac{\bar{y}}{\sqrt{\nu}} \right)) \right] \), where \(\phi(.)\) and \(\Phi(.)\) denote the probability density function (p.d.f.) and cumulative density function (c.d.f.) of standard normal distribution.
Proofs of Lemma 1: (i) For $E(\tilde{R}_j) = \gamma a_j(G_j)\tilde{\xi}_j$ to decrease in $\nu_{\theta_j}$, it is sufficient to show that $a_j(G_j)$ decreases in $\nu_{\theta_j}$. It follows from Proposition 1 and the constructed $G(\nu_{\theta})$ in Eq. (13) that

$$a_j(G_j) = a_j(G(\nu_{\theta})) = \frac{1}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j} - \nu_{\bar{\theta}_j}}(\nu_{\theta_j} - \nu_{\theta}). \tag{30}$$

It follows that

$$\frac{da_j(G_j)}{d\nu_{\theta_j}} \propto \ell(\nu_{\theta}) \equiv \frac{\rho}{\nu_{\theta}^2} - \frac{1 - \rho}{(\nu_{\theta} - \nu_{\theta_j})^2 \nu_{\theta_j} - \nu_{\theta}} \nu_{\theta}. \tag{31}$$

It is obvious after taking derivatives that $\ell(\nu_{\theta})$ decreases in $\nu_{\theta_j}$. Further $\ell(\bar{\nu}_{\theta}) < 0$, and

$$\ell(\nu_{\theta}) \propto \rho - (1 - \rho) \frac{\nu_{\theta}}{\nu_{\theta_j} - \nu_{\theta}} \propto \rho \nu_{\theta_j} - \nu_{\theta} < 0$$

under the assumption $\rho < \min(\frac{\nu_{\theta}}{\nu_{\theta_j} - \nu_{\theta}}, 1 - \frac{\nu_{\theta}}{\nu_{\theta_j} - \nu_{\theta}})$ (see Condition (14)). Taken together, $\ell(\nu_{\theta}) < 0$ and $a_j(G_j)$ decreases in $\nu_{\theta_j}$.

(ii) It suffices to demonstrate that the basic security with a higher $\nu_{\theta_j}$ also has a higher $\text{Cov}(\tilde{R}_j, \tilde{R}_M)$. It follows from Eq. (7) that

$$\frac{d\text{Cov}(\tilde{R}_j, \tilde{R}_M)}{d\nu_{\theta_j}} \propto 1 + 4\gamma^2 a_j(G_j)\nu_{\theta_j}a_j(G_j)^2\ell(\nu_{\theta_j})$$

$$> 1 - 4\gamma^2 a_j(G_j)\nu_{\theta_j}a_j(G_j)^2 \frac{1 - \rho}{(\nu_{\theta_j} - \nu_{\bar{\theta}_j})^2 \nu_{\bar{\theta}_j} - \nu_{\theta}} \nu_{\theta}$$

$$> 1 - a_j(G_j)^2 \frac{1 - \rho}{(\nu_{\theta_j} - \nu_{\bar{\theta}_j})^2 \nu_{\bar{\theta}_j} - \nu_{\theta}} \frac{\nu_{\theta}}{\nu_{\theta_j}}$$

$$= 1 - \frac{1 - \rho}{\left(\frac{\nu_{\theta_j} - \nu_{\bar{\theta}_j}}{\nu_{\theta_j} - \nu_{\theta}}(\nu_{\theta_j} - \nu_{\theta})\right)^2}$$

$$> 1 - \frac{1 - \rho}{\left(\frac{\rho}{\nu_{\theta_j}} + \frac{1 - \rho}{\nu_{\theta_j} - \nu_{\theta}}\right)^2}.$$
where the first inequality follows from the expression of \( \ell(\nu_{\theta_j}) \) (see Eq. 31), the second inequality follows from \( a_j(G_j) < \nu_{\theta_j} \) (see Corollary 1) and the assumption \( \nu_{z_j} < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2) \) (see Condition (2)), and the last inequality obtains under the assumption \( \rho < \min(\frac{\nu_{\theta_j}}{\nu_{\theta_j} - \nu_{\theta_j}}, 1 - \frac{\nu_{\theta_j}}{\nu_{\theta_j} - \nu_{\theta_j}}) \) (see Condition (14)).

Q.E.D.

Proofs of Lemma 2: From Proposition 1, 
\[
\frac{\text{Var}(\tilde{R}_j)}{\text{Var}(R_M)} = \frac{\nu_{\theta_j} + \gamma^2 a_j(G_j)^2 \nu_{z_j}}{\text{Var}(R_M)}.
\]
This implies that typical basic securities with small \( \frac{\nu_{\theta_j}}{\text{Var}(R_M)} \) also have small \( \frac{\nu_{\theta_j}}{\text{Var}(R_M)} \) and \( \gamma^2 a_j(G_j)^2 \nu_{z_j} \). We will use this property in the proof of this Lemma.

From Eq. (15), Proposition 1, and our computation of \( \text{Cov}(\tilde{R}_j, \tilde{R}_M) \) in Appendix B,
\[
\text{IVOL}_{j} = \frac{\text{Var}(\tilde{R}_j) - \text{Cov}(\tilde{R}_j, \tilde{R}_M)^2}{\text{Var}(R_M)} = \nu_{\theta_j} + \gamma^2 a_j(G_j)^2 \nu_{z_j} - \frac{(\nu_{\theta_j} - 2\gamma^2 a_j(G_j)^2 \nu_{z_j})^2}{\text{Var}(R_M)}.
\]
It follows that for the \( j \)'th and \( j' \)'th typical basic securities,
\[
\text{IVOL}_{j} - \text{IVOL}_{j'} = \left[ \nu_{\theta_j} + \gamma^2 a_j(G_{\nu_{\theta_j}})^2 \nu_{z_j} \right] - \left[ \nu_{\theta_{j'}} + \gamma^2 a_j(G_{\nu_{\theta_{j'}}})^2 \nu_{z_{j'}} \right] - \frac{\nu_{\theta_j} + \gamma^2 a_j(G_{\nu_{\theta_j}})^2 \nu_{z_j} + \nu_{\theta_{j'}} + \gamma^2 a_j(G_{\nu_{\theta_{j'}}})^2 \nu_{z_{j'}}}{\text{Var}(R_M)} \left[ \nu_{\theta_j} + \gamma^2 a_j(G_{\nu_{\theta_j}})^2 \nu_{z_j} \right] - \left[ \nu_{\theta_j} + \gamma^2 a_j(G_{\nu_{\theta_j}})^2 \nu_{z_j} \right] - \delta \left[ \nu_{\theta_j} + \gamma^2 a_j(G_{\nu_{\theta_j}})^2 \nu_{z_j} \right] - \left[ \nu_{\theta_{j'}} + \gamma^2 a_j(G_{\nu_{\theta_{j'}}})^2 \nu_{z_{j'}} \right] \approx \nu_{\theta_j} - \nu_{\theta_{j'}} + \gamma^2 \nu_{z_j} \left[ a_j(G_{\nu_{\theta_j}})^2 \right] - a_j(G_{\nu_{\theta_{j'}}})^2 \right],
\]
where \( \delta = \frac{\nu_{\theta j} + 2\gamma^2 a_j(G(\nu_{\theta j}))^2\nu_{zj} + \nu_{\theta j} + 2\gamma^2 a_j(G(\nu_{\theta j}))^2\nu_{zj}}{\text{Var}(\tilde{R}_M)} \epsilon_j^2 \), and the last \( \sim \) follows from the above derived property that for the \( j \)'th and \( j' \)'th typical basic securities, \( \delta \) must be small. For \( \text{IVOL}_j > \text{IVOL}_{j'} \), it suffices that

\[
\frac{d[\nu_{\theta j} + \gamma^2\nu_{zj}a_j(G_j)^2]}{\nu_{\theta j}} \propto 1 + 2\gamma^2 a_j(G_j)\nu_{zj}a_j(G_j)^2 \ell(\nu_{\theta j})
\]

\[
> 1 - 2\gamma^2 a_j(G_j)\nu_{zj}a_j(G_j)^2 \frac{1 - \rho}{(\nu_{\theta j} - \frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2})(\nu_{\theta j} - \nu_{\theta j}^2)} \frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2} - \frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2}
\]

\[
= 1 - \frac{1}{\frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2}(\nu_{\theta j} - \nu_{\theta j}^2)}
\]

\[
> 1 - \frac{1}{1 - \rho \frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2}}
\]

> 0,

where the first inequality follows from the expression of \( \ell(\nu_{\theta j}) \) (see Eq. 31), the second inequality follows from \( a_j(G_j) < \nu_{\theta j} \) (see Corollary 1) and the assumption \( \nu_{zj} < \frac{1}{\gamma^2 \nu_{\theta j}} \min(1/4, \rho/2) \) (see Condition (2)), and the last inequality obtains under the assumption \( \rho < \min(\frac{\nu_{\theta j}}{\nu_{\theta j}}, 1 - \frac{\nu_{\theta j}}{\nu_{\theta j} - \nu_{\theta j}^2}) \) (see Condition (14)).

Q.E.D.

**Proof of Proposition 4:** Taking derivatives of the \( \beta \)-adjusted expected return, \( -\gamma(\nu_{\theta j} + 2\gamma^2 a_j(G_j)^2\nu_{zj} - a_j(G_j))\epsilon_j \), w.r.t. \( \nu_{\theta j} \) yields

\[
\frac{d[-\gamma(\nu_{\theta j} + 2\gamma^2 a_j(G_j)^2\nu_{zj} - a_j(G_j))\epsilon_j]}{d\nu_{\theta j}} \propto -1 + \left[ 1 - 4\gamma^2 a_j(G_j)\nu_{zj} \right] a_j(G_j)^2 \ell(\nu_{\theta j}).
\]

It follows from \( a_j(G_j) < \nu_{\theta j} \) (see Corollary 1) and the assumption \( \nu_{zj} < \frac{1}{\gamma^2 \nu_{\theta j}} \min(1/4, \rho/2) \)
(see Condition (2)) that $1 - 4\gamma^2 a_j(G_j)\nu_{z_j} > 0$. Also, the proof of Lemma 1 shows that under the assumption $\rho < \min\left(\frac{\nu_{\theta}}{\nu_0}, 1 - \frac{\nu_{\theta}}{\nu_0 - \nu_{\theta}}\right)$ (see Condition (14)), $\ell(\nu_{\theta_j}) < 0$. Therefore, the above derivative is negative, and the $\beta$-adjusted expected return decreases in $\nu_{\theta_j}$.

Q.E.D.

**Proof of Proposition 5:** It follows from Eq. (16) that

$$E(\tilde{R}_M) = \sum_{j=1}^{K+N} E\left[\tilde{\xi}_j \tilde{\theta}_j + \tilde{z}_j \tilde{\theta}_j + \gamma a_j(G_j)(\xi_j^2 + 2\tilde{\xi}_j \tilde{z}_j + \tilde{z}_j^2)\right]$$

$$= \sum_{j=1}^{K+N} [\gamma a_j(G_j)(\xi_j^2 + \nu_{z_j})] > 0.$$  

Q.E.D.

**Proof of Proposition 6:** (i) From Eqs. (20) and (21),

$$E(\tilde{R}_{j,\text{NG}}^A) = \gamma \nu_{\theta_j} \tilde{\xi}_j > E(\tilde{R}_{j,\text{G}}^A) = \gamma (\nu_{\theta_j} - G_j/\gamma^2)\tilde{\xi}_j,$$

$$\text{Var}(\tilde{R}_{j,\text{NG}}^A) = \nu_{\theta_j} + \gamma^2 \nu_{\theta_j}^2 \nu_{z_j} > \text{Var}(\tilde{R}_{j,\text{G}}^A) = \nu_{\theta_j} + \gamma^2 (\nu_{\theta_j} - G_j/\gamma^2)^2 \nu_{z_j}.$$  

(ii) From Eq. (20), it follows that the return of the market portfolio in the all-non-$G$ economy is given by

$$\tilde{R}_{M,\text{NG}}^A = \sum_{j=1}^{K+N} (\tilde{\xi}_j + \tilde{z}_j)\tilde{R}_{j,\text{NG}}^A = \sum_{j=1}^{K+N} [(\tilde{\xi}_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma \nu_{\theta_j}(\tilde{\xi}_j + \tilde{z}_j))].$$

It follows immediately that

$$E(\tilde{R}_{M,\text{NG}}^A) = \sum_{j=1}^{K+N} E\left[(\tilde{\xi}_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma \nu_{\theta_j}(\tilde{\xi}_j + \tilde{z}_j))\right] = \sum_{j=1}^{K+N} [\gamma \nu_{\theta_j}(\xi_j^2 + \nu_{z_j})],$$

$$\text{Var}(\tilde{R}_{M,\text{NG}}^A) = \sum_{j=1}^{K+N} \left[(\nu_{\theta_j} + 4\gamma^2 \nu_{\theta_j}^2 \nu_{z_j})\xi_j^2 + \nu_{\theta_j} \nu_{z_j} + 2\gamma^2 \nu_{\theta_j}^2 \nu_{z_j}^2\right],$$

where the last equality follows from Eq. (6) because the all-non-$G$ economy can be viewed as the case in which $a(G_j) = \nu_{\theta_j}$. 

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From Eq. (21), it follows that the return of the market portfolio in the all-$G$ economy is given by

$$\tilde{R}_{M}^{A,G} = \sum_{j=1}^{K+N} (\tilde{\xi}_j + \tilde{z}_j)\tilde{R}_{j}^{A,G} = \sum_{j=1}^{K+N} [(\tilde{\xi}_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma(\nu_{\theta_j} - G_j/\gamma^2)(\tilde{\xi}_j + \tilde{z}_j))].$$

It follows immediately that

$$E(\tilde{R}_{M}^{A,G}) = \sum_{j=1}^{K+N} E[(\tilde{\xi}_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma(\nu_{\theta_j} - G_j/\gamma^2)(\tilde{\xi}_j + \tilde{z}_j))] = \sum_{j=1}^{K+N} \left[\gamma(\nu_{\theta_j} - G_j/\gamma^2)(\tilde{\xi}_j^2 + \nu_{z_j})\right],$$

$$\text{Var}(\tilde{R}_{M}^{A,G}) = \sum_{j=1}^{K+N} \left[(\nu_{\theta_j} + 4\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j})\tilde{\xi}_j^2 + \nu_{\theta_j}\nu_{z_j} + 2\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j}^2\right],$$

where the last equality follows from Eq. (6) because the all-$G$ economy is the case in which $a(G_j) = \nu_{\theta_j} - G_j/\gamma^2$.

A direct comparison indicates that $E(\tilde{R}_{M}^{A,NG}) > E(\tilde{R}_{M}^{A,G})$ and $\text{Var}(\tilde{R}_{M}^{A,NG}) > \text{Var}(\tilde{R}_{M}^{A,G})$.

(iii) Note that $\lambda_{j}^{A,NG} = \frac{\gamma\nu_{\theta_j}\text{Var}(\tilde{R}_{M}^{A,NG})}{\nu_{\theta_j} + 2\gamma^2\nu_{\theta_j}^2\nu_{z_j}}$ and $\lambda_{j}^{A,G} = \frac{\gamma(\nu_{\theta_j} - G_j/\gamma^2)\text{Var}(\tilde{R}_{M}^{A,G})}{\nu_{\theta_j} + 2\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j}}$. From part (ii), $\text{Var}(\tilde{R}_{M}^{A,NG}) > \text{Var}(\tilde{R}_{M}^{A,G})$. Thus, for $\lambda_{j}^{A,NG} > \lambda_{j}^{A,G}$, it suffices that

$$\frac{\nu_{\theta_j}}{\nu_{\theta_j} + 2\gamma^2\nu_{\theta_j}^2\nu_{z_j}} - \frac{\nu_{\theta_j} - G_j/\gamma^2}{\nu_{\theta_j} + 2\gamma^2(\nu_{\theta_j} - G_j/\gamma^2)^2\nu_{z_j}} < 0,$$

where the last inequality obtains under the assumption $\nu_{z_j} < \frac{1}{\gamma^2\nu_{\theta_j}} \min(1/4, \rho/2)$ (see Condition (2)).

Q.E.D.

Proof of Proposition 7: Note that $\lambda_j = \frac{\gamma a_j(G_j)\text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2\nu_{z_j}}$ and $\lambda_j^0 = \frac{(\gamma/\rho)\nu_{\theta_j}\text{Var}(\tilde{R}_M)}{\nu_{\theta_j} + 2(\gamma/\rho)^2\nu_{\theta_j}^2\nu_{z_j}}$. 43
For $\lambda_j < \lambda_j^*$, it suffices to show that

$$\frac{\gamma a_j(G_j)}{\nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_j} - \frac{(\gamma/\rho) \nu_{\theta_j}}{\nu_{\theta_j} + 2(\gamma/\rho)^2 \nu_j^2 \nu_j} \propto \gamma a_j(G_j) \left[ \nu_{\theta_j} + 2(\gamma/\rho)^2 \nu_j^2 \nu_j \right] - (\gamma/\rho) \nu_{\theta_j} \left[ \nu_{\theta_j} + 2\gamma^2 a_j(G_j)^2 \nu_j \right]$$

$$= \left[ \gamma a_j(G_j) - (\gamma/\rho) \nu_{\theta_j} \right] \left[ \nu_{\theta_j} - 2\gamma a_j(G_j)(\gamma/\rho) \nu_{\theta_j} \nu_j \right]$$

$$\propto 2\gamma^2 a_j(G_j) \nu_j - \rho$$

$$< 2\gamma^2 \nu_{\theta_j} \nu_j - \rho$$

$$< 0,$$

where the second “$\propto$” and the first inequality follow from $a_j(G_j) < \nu_{\theta_j}$ (see Corollary 1), and the last inequality obtains under the assumption $\nu_j < \frac{1}{\gamma^2 \nu_{\theta_j}} \min(1/4, \rho/2)$ (see Condition (2)).

Q.E.D.

**Proof of Proposition 8:** Taking the ex ante expectation (over $\tilde{z}_j$) of Eq. (27) yields

$$E^{ex}[U_{G^A}(W_{i1}, X_i)] = E\left[ -\exp\left[ -\gamma M_i - 0.5 \sum_{j=1}^{K+N} \gamma^2 a_j(G_j, \rho_1)^2 (\xi_j + \tilde{z}_j)^2 \nu_j - G_j/\gamma^2 \right] \right]$$

It is straightforward to show, after taking derivatives, that $E^{ex}[U_G(W_{i1}, X)]$ increases in $a_j(G_j, \rho_1)$, and $a_j(G_j, \rho_1)$ decreases in $\rho_1$. Therefore, $E^{ex}[U_G(W_{i1}, X)]$ decreases in $\rho_1$. It is obvious that $E[U_{G^A}(W_{i1}, Y)]$ in Eq. (28) increases in $G^A$.

A $G^A$ trader’s decision to participate in the stock market, instead of gambling through the alternative avenue, is based on

$$\zeta(\rho_1, G^A) \equiv E^{ex}[U_{G^A}(W_{i1}, X_i)] - E[u_{G^A}(W_{i1}, Y)].$$

If $\zeta(\rho_1, \nu_\mu) > 0$, then he will participate the stock market. Otherwise, he will not.

It follows from the above analysis that $\zeta(\rho_1, G^A)$ decreases in $\rho_1$ and $G^A$. Therefore, the unique $\rho_1 \in [0, \rho^A]$ is determined as follows.
• if $\zeta(0, G^A) \leq 0$, then $\rho_1 = 0$;

• if $\zeta(\rho^A, G^A) \geq 0$, then $\rho_1 = \rho^A$;

• if $\zeta(\rho^A, G^A) < 0 < \zeta(0, G^A)$, then an interior $\rho_1$ is specified by $\zeta(\rho_1, G^A) = 0$.

Further, $\rho_1$ decreases in $G^A$.

Q.E.D.

Proof of Corollary 4: It follows from Eqs. (23), (24), (25), and (26) that

$$X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) \sim N(A_{NG}(\rho_1)\bar{\xi}_j, A_{NG}(\rho_1)^2\nu_{z_j}),$$

$$X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j) \sim N(A_G(\rho_1)\bar{\xi}_j, A_G(\rho_1)^2\nu_{z_j}),$$

$$X_{G^A,j}(P_j) \sim N\left(\frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2}\bar{\xi}_j, \left(\frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2}\nu_{z_j}\right)^2\nu_{z_j}\right),$$

where

$$A_{NG}(\rho_1) \equiv \frac{a_j(G_j, \rho_1)}{\nu_{\theta_j}} - 1, \quad \text{and} \quad A_G(\rho_1) \equiv \frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2} - 1.$$

It is easy to show that $A_{NG}(\rho_1) < 0$.

If $A_G(\rho_1) < 0$, then, using the fact listed in Footnote 14, we can express the total expected trading volume in the $j$’th basic security (Eq. 29) as

$$T_j(\rho_1) = 0.5\rho E\left[|X_{NG,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)|\right] + 0.5(1 - \rho)E\left[|X_{G,j}(P_j) - (\bar{\xi}_j + \tilde{z}_j)|\right]$$

$$+ 0.5\rho_1 E\left[|X_{G^A,j}(P_j)|\right]$$

$$= 0.5\rho(-A_{NG}(\rho_1)\sqrt{\nu_{z_j}})\left[2\phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right) - \frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\left(1 - 2\Phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right)\right)\right]$$

$$+ 0.5(1 - \rho)(-A_G(\rho_1)\sqrt{\nu_{z_j}})\left[2\phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right) - \frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\left(1 - 2\Phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right)\right)\right]$$

$$+ 0.5\rho_1\frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2}\sqrt{\nu_{z_j}}\left[2\phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right) + \frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\left(1 - 2\Phi\left(-\frac{\bar{\xi}_j}{\sqrt{\nu_{z_j}}}\right)\right)\right]$$

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\[
\begin{align*}
&= [\rho \cdot \text{NG}(\rho_1) - (1 - \rho) \cdot \text{AG}(\rho_1) + \rho_1 \frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2}] \\
&\quad \times 0.5 \sqrt{\nu_j} \left[2\phi\left(\frac{-\xi_j}{\sqrt{\nu_j}}\right) - \xi_j \sqrt{\nu_j} (1 - 2\Phi\left(\frac{-\xi_j}{\sqrt{\nu_j}}\right))\right].
\end{align*}
\]

Taking derivative w.r.t. \( \rho_1 \) yields
\[
\frac{dT_j(\rho_1)}{d\rho_1} \propto \frac{d}{d\rho_1} \left[\rho \cdot \text{NG}(\rho_1) - (1 - \rho) \cdot \text{AG}(\rho_1) + \rho_1 \frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2}\right] \\
= \frac{a_j(G_j, \rho_1)}{\nu_{\theta_j} - G_j/\gamma^2} + \left(-\frac{\rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{\rho_1}{\nu_{\theta_j} - G_j/\gamma^2}\right) - a_j(G_j, \rho_1)^2 \\
\propto 1 - \left(-\frac{\rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{\rho_1}{\nu_{\theta_j} - G_j/\gamma^2}\right) a_j(G_j, \rho_1) \\
\propto 1 - \left(-\frac{\rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{1 - \rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{\rho_1}{\nu_{\theta_j} - G_j/\gamma^2}\right) \\
\propto 1 - \frac{\rho}{\nu_{\theta_j} - G_j/\gamma^2} + \frac{1 - \rho + \rho_1}{\nu_{\theta_j} - G_j/\gamma^2} \\
> 0.
\]

If \( \text{NG} > 0 \), then we can use a similar analysis to show that \( dT_j(\rho_1)/d\rho_1 > 0 \). Taken together, \( dT_j(\rho_1)/d\rho_1 > 0 \).

Note that \( \rho_1 \) decreases in \( G^\theta \) (see Proposition 8). It follows that \( T_j(\rho_1) \) decreases in \( G^\theta \).

Q.E.D.

**Proof of Proposition 9:** We solve for the equilibrium and prove the proposition using backward induction, in three steps.

**Step 1:** In this step, suppose \( \tilde{\theta}_1 > H_\theta \) and therefore \( P_1 > P_0 \) (which we will show below). An agent remains a non-\( G \) trader (becomes a \( G \) trader) with probability \( \rho (1 - \rho) \). Thus, there is a mass \( \rho \) of non-\( G \) traders and a mass \( 1 - \rho \) of \( G \) traders at Date 2.

Focus on Date 2 for the moment. Write an \( i \)'th non-\( G \) trader’s wealth at Date 3 as
\( W_{i3} = W_{i2} + X_{i2}(V - P_2) \). His expected utility at Date 2 can be expressed as:

\[
E[U_{NG}(W_{i3})|\tilde{\theta}_1, \tilde{\theta}_2] = -\exp\left[ -\gamma W_{i2} - \gamma \left( X_{i2}(E(V|\tilde{\theta}_1, \tilde{\theta}_2) - P_2) - 0.5\gamma X_{i2}^2 \text{Var}(V|\tilde{\theta}_1, \tilde{\theta}_2) \right) \right]. \tag{32}
\]

He needs to choose \( X_{i2} \) to maximize this expected utility. The f.o.c. implies that his demand can be expressed as:

\[
X_{NG,2}(\tilde{\theta}_1, \tilde{\theta}_2, P_2) = \frac{E(V|\tilde{\theta}_1, \tilde{\theta}_2) - P_2}{\gamma \text{Var}(V|\tilde{\theta}_1, \tilde{\theta}_2)} = \frac{\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2}{\gamma \nu_\theta}. \tag{33}
\]

An \( i \)'th G trader's expected utility at Date 2 can be expressed as:

\[
E[U_G(W_{i3}, X_{i2})|\tilde{\theta}_1, \tilde{\theta}_2] = -\exp\left[ -\gamma W_{i2} - \gamma \left( X_{i2}(E(V|\tilde{\theta}_1, \tilde{\theta}_2) - P_2) - 0.5\gamma X_{i2}^2 \text{Var}(V|\tilde{\theta}_1, \tilde{\theta}_2) + 0.5G X_{i2}^2 / \gamma \right) \right]. \tag{34}
\]

He needs to choose \( X_{i2} \) to maximize this expected utility. The f.o.c. implies that his demand can be expressed as:

\[
X_{G2}(\tilde{\theta}_1, \tilde{\theta}_2, P_2) = \frac{E(V|\tilde{\theta}_1, \tilde{\theta}_2) - P_2}{\gamma \text{Var}(V|\tilde{\theta}_1, \tilde{\theta}_2) - G / \gamma} = \frac{\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2}{\gamma (\nu_\theta - G / \gamma^2)}. \tag{35}
\]

It follows from Eqs. (33) and (35) that the market clearing condition requires

\[
\bar{\xi} = \rho \cdot X_{NG,2}(\tilde{\theta}_1, \tilde{\theta}_2, P_2) + (1 - \rho) \cdot X_{G2}(\tilde{\theta}_1, \tilde{\theta}_2, P_2)
= \frac{\rho}{\gamma \nu_\theta} (\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2) + \frac{1 - \rho}{\gamma (\nu_\theta - G / \gamma^2)} (\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2).
\]

Therefore,

\[
P_2 = \bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - \gamma a(G) \bar{\xi},
\]

where \( a(G) = \frac{1}{\rho \frac{1}{\nu_\theta} + \frac{1 - \rho}{\nu_\theta - G / \gamma^2}} \).
Consider a non-$G$ trader’s expected utility at Date 2 in Eq. (32). Plugging in the optimal demand for the risky security from Eq. (33) yields

$$E[U_{NG}(W_{i3})|\tilde{\theta}_1, \tilde{\theta}_2] = -\exp\left[-\gamma W_{i2} - 0.5 \frac{(\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2)^2}{\nu_\theta}\right]$$

$$= -\exp\left[-\gamma W_{i2} - 0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta}\right].$$

(36)

Consider a $G$ trader’s expected utility at Date 2 in Eq. (34). Plugging in the optimal demand for the risky security from Eq. (35) yields

$$E[U_G(W_{i3}, X_{i2})|\tilde{\theta}_1, \tilde{\theta}_2] = -\exp\left[-\gamma W_{i2} - 0.5 \frac{(\bar{V} + \tilde{\theta}_1 + \tilde{\theta}_2 - P_2)^2}{\nu_\theta - G/\gamma^2}\right]$$

$$= -\exp\left[-\gamma W_{i2} - 0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta - G/\gamma^2}\right].$$

(37)

Now focus on Date 1. As argued above, an agent knows that he may or may not become a $G$ trader. Write an $i$’th trader’s wealth at Date 2 as $W_{i2} = W_{i1} + X_{i1}(P_2 - P_1)$. It follows from Eqs. (36) and (37) that his expected utility at Date 1 can be expressed as:

$$E[U(W_{i3})|\tilde{\theta}_1] = \rho E[U_{NG}(W_{i3})|\tilde{\theta}_1] + (1 - \rho)E[U_G(W_{i3}, X_{i2})|\tilde{\theta}_1]$$

$$= -\exp\left[-\gamma W_{i1} - \gamma \left[ X_{i1}(E(P_2|\tilde{\theta}_1) - P_1) - 0.5\gamma X_{i1}^2 \text{Var}(P_2|\tilde{\theta}_1)\right]\right]\left[\rho \exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta}) + (1 - \rho)\exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta - G/\gamma^2})\right].$$

(38)

He needs to choose $X_{i1}$ to maximize this expected utility. The f.o.c. implies that his demand can be expressed as:

$$X_1(\tilde{\theta}_1, P_1) = \frac{E(P_2|\tilde{\theta}_1) - P_1}{\gamma \text{Var}(P_2|\tilde{\theta}_1)} = \frac{\bar{V} + \tilde{\theta}_1 - \gamma a(G)\bar{\xi} - P_1}{\gamma \nu_\theta}.$$

The market clearing condition, $X_1(\tilde{\theta}_1, P_1) = \bar{\xi}$, implies

$$P_1 = \bar{V} + \tilde{\theta}_1 - \gamma \nu_\theta \bar{\xi} - \gamma a(G)\bar{\xi}.$$

(39)
Plugging the derived $X_1(P_1, \tilde{\theta}_1)$ and $P_1$ back into his expected utility at Date 1 using Eq. (38) yields
\[
E\left[ U(W_{i3}) | \tilde{\theta}_1 \right] = -\exp\left[ -\gamma W_{i1} - 0.5 \frac{(\gamma \nu_\theta \xi)^2}{\nu_\theta} \right] \\
\cdot \left[ \rho \exp(-0.5 \frac{(\gamma a(G) \xi)^2}{\nu_\theta}) + (1 - \rho) \exp(-0.5 \frac{(\gamma a(G) \xi)^2}{\nu_\theta - G/\gamma^2}) \right].
\] (40)

**Step 2:** In this step, suppose $\tilde{\theta}_1 \leq H_\theta$ and therefore $P_1 \leq P_0$ (which we will show). An agent remains a non-$G$ trader. We can use a similar derivation as in Step 1, except that we impose $G = 0$ (note $a(0) = \nu_\theta$), to show
\[
P_2 = V + \tilde{\theta}_1 + \tilde{\theta}_2 - \gamma \nu_\theta \xi,
\]
\[
P_1 = \tilde{V} + \tilde{\theta}_1 - 2\gamma \nu_\theta \xi.
\] (41)
Moreover, the agent’s expected utility at date 1 is given by
\[
EU(W_{i3}|\tilde{\theta}_1) = -\exp\left[ -\gamma W_{i1} - \frac{(\gamma \nu_\theta \xi)^2}{\nu_\theta} \right].
\] (42)

**Step 3:** We now focus on date 0. Since it is before $\tilde{\theta}_1$ is released and $P_1$ is formed, all agents have identical preferences and beliefs and hold the same long position of the risky asset to clear the market.

Write an $i$‘th trader’s wealth at date 1 as $W_{i1} = W_{i0} + X_{i0}(P_1 - P_0)$. If $\tilde{\theta}_1 > H_\theta$ so $P_1 > P_0$ and he makes money, he may become a $G$ trader. He will be in the regime characterized by Eqs. (39) and (40). If $\tilde{\theta}_1 \leq H_\theta$ so $P_1 \leq P_0$ and he does not make money, he will remain a non-$G$ trader for sure. He will be in the regime characterized by Eqs. (41), and (42). Accounting for both cases, we can write his expected utility at date 0 as
\[
E\left[ U(W_{i3}) \right] = -\int_{-\infty}^{\infty} \exp\left[ -\gamma W_{i0} - \gamma X_{i0}(V + \tilde{\theta}_1 - \gamma \nu_\theta \xi - \gamma a(G) \xi - P_0) - 0.5 \frac{(\gamma \nu_\theta \xi)^2}{\nu_\theta} \right] \\
\cdot \left[ \rho \exp(-0.5 \frac{(\gamma a(G) \xi)^2}{\nu_\theta}) + (1 - \rho) \exp(-0.5 \frac{(\gamma a(G) \xi)^2}{\nu_\theta - G/\gamma^2}) \right] d\Phi\left( \frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}} \right) \\
- \int_{H_\theta}^{\infty} \exp\left[ -\gamma W_{i0} - \gamma X_{i0}(V + \tilde{\theta}_1 - 2\gamma \nu_\theta \xi - P_0) - \frac{(\gamma \nu_\theta \xi)^2}{\nu_\theta} \right] d\Phi\left( \frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}} \right),
\]
where $\Phi(\cdot)$ is the cumulative density function of standard normal distribution. The agent needs to choose $X_{i0}$ to maximize this expected utility. Taking the f.o.c. w.r.t. $X_{i0}$, imposing the market clearing condition $X_{i0} = \bar{\xi}$, and simplifying items yields

$$0 = \int_{H_\theta}^{\infty} \frac{\tilde{V} + \tilde{\theta}_1 - \gamma \nu_\theta \bar{\xi} - \gamma a(G)\bar{\xi} - P_0}{\exp(\gamma \bar{\xi} \tilde{\theta}_1)} \cdot \left[ \rho \exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta}) + (1 - \rho) \exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta - G/\gamma^2}) \right] d\Phi(\frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}})
+ \int_{-\infty}^{H_\theta} \frac{\tilde{V} + \tilde{\theta}_1 - 2 \gamma \nu_\theta \bar{\xi} - P_0}{\exp[\gamma \bar{\xi}(\tilde{\theta}_1 - \gamma (\nu_\theta - a(G))\bar{\xi})]} \exp(-0.5 \frac{(\gamma \nu_\theta \bar{\xi})^2}{\nu_\theta}) d\Phi(\frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}}).$$

(43)

If $\tilde{\theta}_1 = H_\theta$, then $P_1 = P_0$ where $P_1$ is given by Eq. (41). This implies

$$P_0 = \tilde{V} + H_\theta - 2 \gamma \nu_\theta \bar{\xi}.$$ 

Plugging this expression for $P_0$ back into Eq. (43) yields

$$0 = \int_{H_\theta}^{\infty} \frac{\tilde{\theta}_1 - H_\theta + \gamma (\nu_\theta - a(G))\bar{\xi}}{\exp(\gamma \bar{\xi} \tilde{\theta}_1)} \cdot \left[ \rho \exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta}) + (1 - \rho) \exp(-0.5 \frac{(\gamma a(G)\bar{\xi})^2}{\nu_\theta - G/\gamma^2}) \right] d\Phi(\frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}})
+ \int_{-\infty}^{H_\theta} \frac{\tilde{\theta}_1 - H_\theta}{\exp[\gamma \bar{\xi}(\tilde{\theta}_1 - \gamma (\nu_\theta - a(G))\bar{\xi})]} \exp(-0.5 \frac{(\gamma \nu_\theta \bar{\xi})^2}{\nu_\theta}) d\Phi(\frac{\tilde{\theta}_1}{\sqrt{\nu_\theta}}).$$

It is straightforward to show that the right hand side of this equation decreases in $H_\theta$, and is positive (negative) if $H_\theta \searrow -\infty$ ($H_\theta \nearrow \infty$). Therefore, $H_\theta$ is uniquely determined by this equation.

Q.E.D.
Appendix B

[Computation of Market Portfolio’s Return Volatility From the Definition in Section 2.3)]

From Proposition 1, the returns of the $j$’th basic security and the market portfolio are given by

\[
\tilde{R}_j = \tilde{\theta}_j - P_j = \tilde{\theta}_j + \gamma a_j(G_j)(\xi_j + \tilde{z}_j),
\]

\[
\tilde{R}_M = \sum_{j=1}^{K+N} (\xi_j + \tilde{z}_j) \tilde{R}_j = \sum_{j=1}^{K+N} \left[ (\xi_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma a_j(G_j)(\xi_j + \tilde{z}_j)) \right].
\]

It follows that

\[
\text{Var}(\tilde{R}_M) = \sum_{j=1}^{K+N} \text{Var}\left[ (\xi_j + \tilde{z}_j)(\tilde{\theta}_j + \gamma a_j(G_j)(\xi_j + \tilde{z}_j)) \right]
\]

\[
= \sum_{j=1}^{K+N} \text{Var}\left[ \xi_j \tilde{\theta}_j + 2\xi_j \gamma a_j(G_j)\tilde{z}_j + \tilde{z}_j \tilde{\theta}_j + \gamma a_j(G_j)\tilde{z}_j^2 \right]
\]

\[
= \sum_{j=1}^{K+N} \left[ E\left[ (\xi_j \tilde{\theta}_j + 2\xi_j \gamma a_j(G_j)\tilde{z}_j + \tilde{z}_j \tilde{\theta}_j + \gamma a_j(G_j)\tilde{z}_j^2)^2 \right] \right.
\]

\[
- \left. E\left( \xi_j \tilde{\theta}_j + 2\xi_j \gamma a_j(G_j)\tilde{z}_j + \tilde{z}_j \tilde{\theta}_j + \gamma a_j(G_j)\tilde{z}_j^2 \right)^2 \right]
\]

\[
= \sum_{j=1}^{K+N} \left[ (\nu_{\theta_j} + 4\gamma^2 a_j(G_j)^2 \nu_{z_j}) \xi_j^2 + \nu_{\theta_j} \nu_{z_j} + 3\gamma^2 a_j(G_j)^2 \nu_{z_j}^2 - \gamma^2 a_j(G_j)^2 \nu_{z_j}^2 \right]
\]

\[
= \sum_{j=1}^{K+N} \left[ (\nu_{\theta_j} + 4\gamma^2 a_j(G_j)^2 \nu_{z_j}) \xi_j^2 + \nu_{\theta_j} \nu_{z_j} + 2\gamma^2 a_j(G_j)^2 \nu_{z_j}^2 \right].
\]

Note that in the third equality, all cross-products of items in the first bracket have mean zero. In the fourth equality, $E(\tilde{z}_j^4) = 3\nu_{z_j}^2$.

Q.E.D.
[Computation of the Covariance between the j’th Basic Security’s and Market Portfolio’s Returns From the Definition in Section 2.4]

From the expressions of $\tilde{R}_j$ and $\tilde{R}_M$,

$$\text{Cov}(\tilde{R}_j, \tilde{R}_M) = \text{Cov}(\tilde{R}_j, \sum_{j=1}^{K+N} (\tilde{\xi}_j + \tilde{z}_j)\tilde{R}_j)$$

$$= \text{Cov}(\tilde{R}_j, (\tilde{\xi}_j + \tilde{z}_j)\tilde{R}_j) = \text{Cov}(\tilde{R}_j, \tilde{\xi} \tilde{R}_j) + \text{Cov}(\tilde{R}_j, \tilde{z}_j \tilde{R}_j),$$

where the second equality obtains because basic securities have independent random supplies and payoffs.

$$\text{Cov}(\tilde{R}_j, \tilde{\xi} \tilde{R}_j) = \tilde{\xi}_j \text{Var}(\tilde{R}_j) = \nu_0 \tilde{\xi}_j + \gamma^2 a_j (G_j)^2 \nu_z \tilde{\xi}_j,$$

$$\text{Cov}(\tilde{R}_j, \tilde{z}_j \tilde{R}_j) = E(\tilde{z}_j \tilde{R}_j^2) - E(\tilde{R}_j)E(\tilde{z}_j \tilde{R}_j)$$

$$= E[\tilde{z}_j (\tilde{\theta}_j + \gamma a_j (G_j)(\tilde{\xi}_j + \tilde{z}_j))^2]$$

$$- E[\tilde{\theta}_j + \gamma a_j (G_j)(\tilde{\xi}_j + \tilde{z}_j)]E[\tilde{z}_j (\tilde{\theta}_j + \gamma a_j (G_j)(\tilde{\xi}_j + \tilde{z}_j))]$$

$$= 2\gamma^2 a_j (G_j)^2 \nu_z \tilde{\xi}_j - \gamma^2 a_j (G_j)^2 \nu_z \tilde{\xi}_j$$

$$= \gamma^2 a_j (G_j)^2 \nu_z \tilde{\xi}_j.$$

Therefore, $\text{Cov}(\tilde{R}_j, \tilde{R}_M) = \nu_0 \tilde{\xi}_j + 2\gamma^2 a_j (G_j)^2 \nu_z \tilde{\xi}_j$.

Q.E.D.
References


Luo, Jiang and Avanidhar Subrahmanyam (2016), “Financial market equilibrium when information is asymmetric and stock ownership is a consumption good,” *UCLA working paper*.


Figure 1: Price Reaction and Long-Run Performance

This graph plots the price paths conditional on the public announcement $\tilde{\theta}_1$. We assume the parameter values $\bar{V} = 5$, $\nu_\theta = 1$, $\xi = 1$, $\gamma = 0.5$, $\rho = 0.5$, and $G = 0.2$. This implies that the threshold for the probabilistic conversion to a $G$ trader is $H_\theta = -0.388$. The realizations of $\tilde{\theta}_2$ and $\tilde{\theta}_3$ are assumed to be zero, i.e., their mean. From the bottom to the top, each path represents a realization of $\tilde{\theta}_1$ from -1 to 0 (step size=0.025). $\tilde{\theta}_1 \leq H_\theta$ for the paths indicated by $\triangle$'s. $\tilde{\theta}_1 > H_\theta$ for the paths indicated by *'s.