Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps

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We develop a model of nominal and real bond yield curves that has four stochastic drivers but seven factors: three factors primarily determine the cross-section of yields, whereas four volatility factors solely determine risk premia. The model is estimated using nominal Treasury yields, survey inflation forecasts, and inflation swap rates and has attractive empirical properties. Time-varying volatility is particularly apparent in short-term real rates and expected inflation. Also, we detail the different economic forces that drive short- and long-term real and inflation risk premia and provide evidence that Treasury inflation-protected securities were undervalued prior to 2004 and during the recent financial crisis. (JEL G01, G12, G13)

Policymakers and finance professionals often use the term structure of Treasury yields to infer expectations of inflation and real interest rates. Inflation expectations can gauge the credibility of a government’s fiscal and monetary policies, whereas real rates measure the economic cost of financing investments and the tightness of monetary policy. However, Treasury yields embed time-varying risk premia that make it difficult to extract measures of expected inflation and real rates. This article presents a new methodology for decomposing Treasury yields and analyzes the determinants of the term structures of real rates, expected inflation, and inflation risk premia. The article

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has two main innovations relative to the existing literature. First, it develops a new model of nominal and real term structures that provides a convenient, yet realistic, framework for identifying the dynamics of real and inflation-related factors. Second, the article introduces a new data source for estimating such models, namely, zero-coupon inflation swaps. We present evidence that the difference between nominal yields and inflation swap rates provides more reliable information on real yields than do inflation-indexed bond yields.

Our model of nominal and real term structures has several desirable properties. It generates simple, closed-form solutions for nominal yields, inflation-indexed yields, inflation swap rates, and expected inflation that make empirical estimation straightforward. Furthermore, it is consistent with two important empirical properties of nominal yields: yields have stochastic volatilities that are correlated with their levels (Ait-Sahalia 1996; Brenner, Harjes, and Kroner 1996; Gallant and Tauchen 1998); and risk premia (expected excess returns) on longer-maturity bonds are highest when the yield curve is steep (Fama and Bliss 1987; Campbell and Shiller 1991). Since it captures the relevant empirical features of nominal yields, the model is likely to provide a correct starting point for decomposing these yields into real rates, expected inflation, and risk premia.

It may be surprising that our term structure model is in the completely affine class. Dai and Singleton (2000) show that many completely affine models fail to possess important empirical features of nominal yields. However, our model is outside the subclass that they studied and differs because it has four stochastic drivers (sources of risk) yet seven state variables. Three of the state variables are the short-term real interest rate, expected inflation, and inflation’s “central tendency,” and they have a large influence on the cross-section of bond yields. However, they play no direct role in determining bonds’ risk premia. Rather, bond risk premia depend on four volatility-state variables that have dynamics driven by normal and chi-squared innovations that derive from inflation and the aforementioned three state variables. This decoupling of the state variables that largely determine the cross-section of yields, versus those that solely determine risk premia, allows for time-varying risk premia that can even change sign. As a result, the model’s ability to fit the cross-section and time-series of yields exceeds that of traditional affine models.

The article’s second main innovation is its use of data on inflation swap rates, in addition to nominal Treasury yields and survey forecasts of inflation,

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1 In response to limitations of completely affine models, Duffee (2002) and Dai and Singleton (2000) develop more general “essentially affine” models that permit time-varying risk premia capable of producing positive correlation between bond excess returns and the slope of the yield curve. This feature occurs when the models’ state variables are Gaussian, but such models cannot capture the equally well-established time variation in yields’ volatilities that are positively related to their levels.

2 Different from the usual completely affine models, our model has market prices of risk that are neither constant, as in Vasicek (1977), nor proportional to the square root of variables that directly affect the levels of yields, as in Cox, Ingersoll, and Ross (1985).
to estimate the model’s parameters. Identifying the parameters of a joint model of nominal and real term structures requires more than just data on nominal yields. Previous work using U.S. data typically employs nominal Treasury yields along with either Treasury inflation-protected securities (TIPS) yields (D’Amico, Kim, and Wei 2008; Chen, Liu, and Cheng 2010; Christensen, Lopez, and Rudebusch 2010) or survey forecasts of inflation (Pennacchi 1991; Chernov and Mueller, forthcoming).³,⁴ Unlike most studies, we use three different sources of data. We show that inflation-indexed yields can be computed as the difference between equivalent maturity nominal Treasury yields and inflation swap rates, and these derived real yields are less prone to liquidity shocks than are TIPS yields. Consequently, model estimation using inflation swap rates, rather than TIPS yields, leads to more reliable parameter estimates. Whereas using survey inflation forecasts, in addition to nominal yields and inflation swaps, creates extra demands on our model’s ability to match all observations, it permits better identification of physical expectations of inflation (as reflected in survey forecasts) versus inflation risk premia (as reflected in nominal yields).

Based on the estimated model, we are able to compute term structures of inflation expectations and inflation-indexed (real) yields over our entire 1982-to-2010 sample period. Comparing our model-implied real yields to TIPS yields beginning in 1999, we confirm the results of prior studies that found massive underpricing of TIPS during their early years, followed by fair pricing from 2004 to 2008. Enormous underpricing of TIPS reappeared during the 2008-to-2009 financial crisis years.

We obtain several other noteworthy results. First, we find that the short-term real interest rate is typically the most volatile component of the yield curve, and it is especially important to allow its volatility to be stochastic. Real rates were negative for much of 2002 to 2005, which may have helped inflate a credit bubble. Second, we find that expected inflation over short horizons is also volatile and has high negative correlation with real rates, likely an artifact of the Federal Reserve’s policy of pegging short-term nominal interest rates. Moreover, both real rates and expected inflation display strong mean reversion. Third, over our 1982-to-2010 sample period, inflation’s central tendency, which can be viewed as investors’ expectation of longer-term inflation, declined substantially, consistent with greater credibility of the Federal Reserve’s desire to maintain low inflation.


⁴ Ang, Bekaert, and Wei (2007) develop a regime-switching model that is estimated using data on nominal yields and actual inflation. Identification is achieved by inferring expected inflation from the actual inflation process along with imposing other parameter restrictions. Similarly, Buraschi and Jiltsov (2005) develop a structural monetary model, whose parameters are estimated using data on nominal yields and the processes for actual inflation and the M2 money supply.
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Our last results explain the term structures of real and inflation risk premia, which are solely determined by the model’s four volatility-state variables. Real risk premia averaged 25, 57, and 103 basis points for two-, five-, and ten-year maturities, respectively. We show that shorter-maturity real risk premia increase when the volatility of the short-term real rate is high. However, increases in longer-maturity real risk premia occur mainly with a rise in the volatility of inflation’s central tendency, reflecting possible nonneutrality in longer-run inflation. We also estimate an inflation risk premium that averaged −5, 17, and 44 basis points for two-, five-, and ten-year maturities. At short maturities, inflation risk premia are inversely related to the volatility of unanticipated inflation or deflation. This volatility factor also reduces longer-maturity inflation risk premia during financial crises when there is a “flight-to-quality.” During normal times, however, longer-maturity inflation risk premia are more positively related to the volatility of inflation’s central tendency. These results accord with economic intuition regarding how different sources of real and inflation uncertainty affect the term structure of risk premia.

The article proceeds as follows. Section 1 introduces a model of real interest rates and inflation that is used to derive the term structures of nominal bonds, inflation forecasts, inflation-indexed bonds, and inflation swap rates. Section 2 describes the data used and explains the estimation technique. Section 3 describes the results, and Section 4 concludes.

1. A Model of Nominal and Real Term Structures

Consider a discrete time environment with multiple periods, each of length $\Delta t$ measured in years. Let $M_t$ be the nominal pricing kernel with dynamics

$$
\frac{M_{t+\Delta t}}{M_t} = e^{-i_t \Delta t - \frac{1}{2} \sum_{j=1}^{4} \phi_j^2 h_{j,t} \Delta t - \sum_{j=1}^{4} \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t}}.
$$

(1)

Here, $\epsilon_{j,t+\Delta t}$, $j = 1, 2, \ldots, 4$ are independent standard normal random variables and $\phi_j h_{j,t}$, $j = 1, 2, \ldots, 4$ are market prices of risk associated with these four sources of uncertainty. The $\phi_j$ are constants, whereas the $h_{j,t}$ are volatility-state variables whose dynamics will be specified shortly. $i_t$ is the annualized, one-period nominal interest rate.

Denote the consumer price index (dollar value of the consumption basket) at date $t$ as $I_t$. Its dynamics are

$$
\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t + h_{1,t} \sqrt{\Delta t} \epsilon_{1,t+\Delta t}},
$$

(2)

where $\pi_t = \frac{1}{\Delta t} \ln \left( E_t \left[ I_{t+\Delta t}/I_t \right] \right)$ is the rate of expected inflation from $t$ to $t + \Delta t$. 

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Therefore, the process for the real (inflation-indexed) pricing kernel, $m_t$, is

$$\frac{m_{t+\Delta t}}{m_t} = \frac{M_{t+\Delta t}}{M_t} \frac{I_{t+\Delta t}}{I_t}$$  \hspace{1cm} (3)

$$= e^{\left(\pi_t - i_t - \frac{1}{2} h_{j,t}^2 \right) \Delta t - \frac{1}{2} \sum_{j=1}^{4} \phi_j^2 h_{j,t}^2 \Delta t - \sum_{j=1}^{4} \phi_j h_{j,t} \sqrt{\Delta t} \epsilon_{j,t+\Delta t} + h_{1,t} + \sqrt{\Delta t} \epsilon_{1,t+\Delta t}}.$$  \hspace{1cm} (4)

Taking expectations on the left-hand side of Equation (3) defines $r_t$, the one-period real rate:

$$E_t \left[ \frac{m_{t+\Delta t}}{m_t} \right] = e^{-r_t \Delta t}.$$  \hspace{1cm} (5)

Taking expectations on the right-hand side of Equation (4) implies that

$$i_t = \pi_t + r_t - \phi_1 h_{1,t}^2.$$  \hspace{1cm} (6)

To complete the model, the dynamics of the state variables are specified as

$$\pi_{t+\Delta t} - \pi_t = [a_t + a_1 r_t + a_2 \pi_t] \Delta t + \sqrt{\Delta t} \sum_{j=1}^{4} \beta_j h_{j,t} \epsilon_{j,t+\Delta t}$$

$$r_{t+\Delta t} - r_t = [b_0 + b_1 r_t + b_2 \pi_t] \Delta t + \sqrt{\Delta t} \sum_{j=1}^{4} \gamma_j h_{j,t} \epsilon_{j,t+\Delta t}$$

$$a_{t+\Delta t} - a_t = [c_0 + c_1 a_t] \Delta t + \sqrt{\Delta t} \sum_{j=1}^{4} \rho_j h_{j,t} \epsilon_{j,t+\Delta t}$$

$$h_{j,t+\Delta t} - h_{j,t}^2 = [d_{j0} + d_{j1} h_{j,t}^2] \Delta t + d_{j2} \Delta t (\epsilon_{j,t+\Delta t} - d_{j3} h_{j,t}^2)^2, \hspace{0.5cm} j = 1, \ldots, 4,$$  \hspace{1cm} (7)

where $a_t$ is an additional state variable that shifts the future path of $\pi_t$. Subject to stationarity conditions, the unconditional means (steady-state levels) of $\pi_t$ and $r_t$ are

$$\overline{\pi} = -\frac{a_1 b_0 c_1 + b_1 c_0}{(a_1 b_2 - a_2 b_1) c_1},$$  \hspace{1cm} (8)

$$\overline{r} = \frac{a_2 b_0 c_1 + b_2 c_0}{(a_1 b_2 - a_2 b_1) c_1}.$$

The unconditional mean of $a_t$ is $-c_0/c_1 = -(a_1 \overline{r} + a_2 \overline{\pi})$. If a constant is added to $a_t$ such that $\overline{a}_t \equiv a_t + a_1 \overline{r} + (1 + a_2) \overline{\pi},$ then the unconditional mean of $\overline{a}_t$ equals $\overline{\pi}$, and $\overline{a}_t$ is commonly referred to as the “central tendency” of the rate of expected inflation. For simplicity, we refer to $a_t$ as the central tendency, but it should be understood that it differs from the true central tendency, $\overline{a}_t$, by a constant.

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5 Prior research supports a time-varying central tendency for inflation in order to adequately fit the term structure (Balduzzi, Das, and Foresi 1998; Jegadeesh and Pennacchi 1996). Kozicki and Tinsley (2009) show that a “shifting endpoint” for the short-term interest rate process captures historical changes in market perceptions of the policy target for inflation and significantly improves long-horizon forecasts of short-term interest rates.
The \( h_{j,t} \) are volatility-state variables that satisfy the nonlinear asymmetric GARCH model of Engle and Ng (1993). Subject to stationarity conditions, their steady states are

\[
\tilde{h}_j^2 = -\frac{d_{j0} + d_{j2}}{d_{j1} + d_{j2}d_{j3}^2}, \quad j = 1, \ldots, 4. \tag{9}
\]

Equations (2) and (6) specify that actual inflation, expected inflation, the real interest rate, and inflation’s central tendency follow imperfectly correlated, stochastic volatility processes. Their correlations depend on the \( \beta_j, \gamma_j, \) and \( \rho_j \) coefficients multiplying the four orthogonal shocks, \( h_{j,t}, \epsilon_{j,t+\Delta t}, j = 1, \ldots, 4, \) but without loss of generality, we can restrict \( \beta_2 = \gamma_3 = \rho_4 = 1. \) If the four volatility-state variables are shut down, the model becomes Gaussian in the three state variables, \( \pi_t, r_t, \) and \( \alpha_t. \)

The model’s state variables can be written as a \( 7 \times 1 \) vector \( x_t \equiv (\pi_t, r_t, \alpha_t, h_{1t}^2, h_{2t}^2, h_{3t}^2, h_{4t}^2)' \), whereas the market prices of risk associated with each of the four shocks \( h_{j,t}, \epsilon_{j,t+\Delta t}, j = 1, \ldots, 4 \) are the \( 4 \times 1 \) vector \( \Lambda_t \equiv (\phi_1 h_{1t}, \phi_2 h_{2t}, \phi_3 h_{3t}, \phi_4 h_{4t})' \). The compensation for risk depends on the square roots of the \( h_{j,t}^2 \) state variables but not the other state variables. Furthermore, because the processes for the \( h_{j,t}^2 \) state variables depend on both the levels of the innovations, \( \epsilon_{j,t}, \) and their squares, \( \epsilon_{j,t}^2, \) the date \( t + \Delta t \) distribution of the state vector \( x_{t+\Delta t} \) conditional on \( x_t \) is not multivariate normal but a mixture of normals and chi-squared distributions. Since bond yields are shown to be affine in \( x_t, \) yield changes will display the skewness and kurtosis derived from \( x_t. \)

As \( \Delta t \to 0, \) our model can be made to converge to many possible diffusive limits. The proposition below describes one possible case.

**Proposition 1.** If we define \( d_{j0} = (\kappa_j \theta_j - \frac{\nu_j^2}{4}), d_{j1} = -\frac{1}{\Delta t}, d_{j2} = \frac{\nu_j^2}{4}, \) and \( d_{j3} = \frac{2(1-\kappa_j \Delta t/2)}{\nu_j \sqrt{\Delta t}}, \) then the limiting dynamics of Equation (6) is

\[
\begin{align*}
    d\pi_t &= (a_t + a_1 r_t + a_2 \pi_t)dt + \sum_{j=1}^2 \beta_j h_{j,t}dW_j(t) \\
    dr_t &= (b_0 + b_1 r_t + b_2 \pi_t)dt + \sum_{j=1}^3 \gamma_j h_{j,t}dW_j(t) \\
    d\alpha_t &= (c_0 + c_1 \alpha_t)dt + \sum_{j=1}^4 \rho_j h_{j,t}dW_j(t) \\
    dh_{j,t}^2 &= \kappa_j (\theta_j - h_{j,t}^2)dt - \nu_j \sqrt{h_{j,t}^2}dW_j(t), \quad j = 1, \ldots, 4, \tag{10}
\end{align*}
\]

where \( dW_{jt}, j = 1, \ldots, 4 \) are independent Wiener processes.

**Proof.** See Appendix.

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6 This “no GARCH” case occurs when \( d_{j1} = -1/\Delta t \) and \( d_{j2} = d_{j3} = 0, \) so that \( h_{j,t}^2 = \tilde{h}_j^2 \forall t. \)
Whereas our empirical model assumes a period of one month ($\Delta t = 1/12$), it may be interpreted as approximating the above continuous-time, stochastic volatility model.

1.1 Nominal and inflation-indexed bonds

The model is estimated using data on nominal Treasury yields, inflation swap rates, and survey forecasts of inflation. This section derives model prices for nominal and inflation-indexed (real) bonds, which also determine inflation swap rates. The following section gives the model formula for inflation forecasts.

Let $P^{(n)}_{N,t}$ and $y^{(n)}_{N,t}$ be the date $t$ price and continuously compounded yield, respectively, of a nominal bond that pays one dollar at date $t + n\Delta t$. Similarly, $P^{(n)}_{R,t}$ and $y^{(n)}_{R,t}$ are the date $t$ real price and yield of a bond that pays one unit of the consumption basket at date $t + n\Delta t$. In practice, an inflation-indexed bond typically pays semi-annual coupons, but since its payments are a portfolio of zero-coupon payments, it is sufficient to value a zero-coupon inflation-indexed bond. Moreover, inflation-indexed payments are not fully protected against inflation. For example, the inflation index for a TIPS payment is based on the consumer price index (CPI) recorded three months prior to the payment date.\(^7\)

Accounting for this indexation lag, define $V^{(n,d)}_{t,t_{s}}$ as the date $t$ nominal price of receiving $I_{t_{e}}/I_{t_{s}}$ dollars at date $t_{p}$, with $n$ periods to go to date $t_{p}$, and let $d$ be the indexation lag in periods. Following actual practice, the payment made at date $t_{p}$ is based on accumulated inflation from dates $t_{s} \equiv t_{0} - d\Delta t$ to $t_{e} \equiv t_{p} - d\Delta t$. Thus, the payment at date $t_{p}$ equals accumulated inflation, $I_{t_{e}}/I_{t_{s}}$, over the life of the bond but lagged $d$ periods. For TIPS, $d\Delta t = 3 \times 1/12 = 1/4$ year. Note that at date $t_{e}$, the value of the payment made $d$ periods later is

$$V^{(d,d)}_{t_{e},t_{s}} = \frac{I_{t_{e}}}{I_{t_{s}}} P^{(d)}_{N,t_{e}}, \quad (11)$$

and at date $t$, with $n$ periods to go to date $t_{p}$, we have

$$V^{(n,d)}_{t,t_{s}} = E_t \left[ \frac{M_{t+\Delta t}}{M_t} V^{(n-1,d)}_{t+\Delta t,t_{s}} \right]. \quad (12)$$

Since $V^{(n,d)}_{t,t_{s}}$ is the date $t$ nominal price of receiving $I_{t_{e}}/I_{t_{s}}$ dollars at date $t_{p}$, $V^{(n,d)}_{t,t_{s}} I_{t_{s}}$ is the nominal price of receiving $I_{t_{e}}$ dollars at date $t_{p}$. If $P^{(n,d)}_{R,t}$ is defined to be the date $t$ real price of receiving $I_{t_{e}}$ dollars at date $t_{p}$, then

$$P^{(n,d)}_{R,t} = V^{(n,d)}_{t,t_{s}} \frac{I_{t_{e}}}{I_{t}}. \quad (13)$$

\(^7\) A reason for this delay is that the CPI is reported with a lag after the date for which it is recorded. Most models ignore this indexation lag, though Risa (2001) is an exception.
With no indexation delay \((d = 0)\), \(P^{(n,0)}_{R,t} = P^{(n)}_{R,t}\) represents the real price of a claim that pays one unit of the consumption basket in \(n\) periods.

**Inflation Swaps**

“Zero-coupon inflation swaps” are the most liquid of all the over-the-counter market inflation derivative products. They are quoted with maturities ranging from one to 30 years. Together with nominal Treasuries, they provide an alternative measure of real yields.

A zero-coupon inflation swap is a forward contract, whereby the inflation buyer pays a predetermined fixed nominal rate and in return receives from the seller an inflation-linked payment. Denote the swap’s initiation date as \(t_0\) and its maturity (payment) date as \(t_p\). Similar to TIPS, the inflation-linked payment made at date \(t_p\) equals \(I_{t_e}/I_{t_s}\), where, as before, \(ts = t_0 - d\Delta t\), \(te = t_p - d\Delta t\), and \(d\Delta t = \frac{1}{4}\) years. In return for receiving \(I_{t_e}/I_{t_s}\), the inflation buyer makes a fixed payment of \(e^{k(t_e-ts)}\), where \(k\) is the continuously compounded inflation swap rate. Thus, the net fixed for inflation swap payment is \(e^{k(t_e-ts)} - I_{t_e}/I_{t_s}\).

Viewed from date \(t\), the value of the fixed (nominal) leg is simply

\[
V_{fix}(t) = P^{(n)}_{N,t} e^{k(t_e-ts)}. \tag{14}
\]

The value of the inflation leg, say \(V_{inf}(t)\), equals the value of a zero-coupon TIPS with payouts at date \(t_p\) linked to the index values at dates \(t_s\) and \(t_e\):

\[
V_{inf}(t) = V^{(n,d)}_{t,t_s} = P^{(n,d)}_{R,t} \frac{I_{t_e}}{I_{t_s}}. \tag{15}
\]

At the initiation date, \(t_0\), the fair swap rate is that which equates \(V_{fix}(t_0)\) to \(V_{inf}(t_0)\):

\[
k^*(t_0; t_s, t_e) = y^{(n)}_{N,t_0} - y^{(n,d)}_{R,t_0} = be^{(n,d)}_{t_0}, \tag{16}
\]

where \(y^{(n,d)}_{R,t_0}\) is defined as

\[
y^{(n,d)}_{R,t_0} = -\frac{1}{n\Delta t} \ln V^{(n,d)}_{t_0,t_s} = -\frac{1}{n\Delta t} \ln \left( P^{(n,d)}_{R,t_0} I_{t_0}/I_{t_s} \right), \tag{17}
\]

and \(be^{(n,d)}_{t}\) is the date \(t\) break-even inflation rate for a maturity of \(n\Delta t\) years. The above shows that once we have a valuation equation for a TIPS, we also have a value for a fair inflation swap rate. Moreover, \(y^{(n)}_{N,t_0} - k^*(t_0; t_s, t_e) = y^{(n,d)}_{R,t_0}\) is a measure of an \(n\)-period maturity real yield that is an alternative to a TIPS yield.

The following proposition provides the recursive equations for both nominal and real (e.g., TIPS) bond values, which in turn can be used to value inflation swaps.

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8 In practice, inflation swap rates are quoted as annually compounded rates, say \(k_a\), where \(k_a = \ln(k) - 1\). We translate these rates to continuously compounded ones.
Proposition 2. Under the above dynamics, nominal and real bond prices are given by

\[ P_{N,t}^{(n)} = e^{-K_n - A_n \pi_t - B_n r_t - C_n a_t - \sum_{j=1}^{4} D_{j,n} h_{j,t}^2} \quad \text{for } n \geq 1, \quad (18) \]

\[ P_{R,t}^{(n,d)} = e^{-\tilde{K}_n - \tilde{A}_n \pi_t - \tilde{B}_n r_t - \tilde{C}_n a_t - \sum_{j=1}^{4} \tilde{D}_{j,n} h_{j,t}^2} \quad \text{for } n \geq d, \quad (19) \]

where \( K_1 = 0, A_1 = \Delta t, B_1 = \Delta t, C_1 = 0, D_{1,1} = -\phi_1 \Delta t, D_{j,1} = 0 \) for \( j = 2, 3, 4, \) and \( \tilde{K}_d = K_d, \tilde{A}_d = A_d, \tilde{B}_d = B_d, \tilde{C}_d = C_d, \tilde{D}_{j,d} = D_{j,d} \) for \( j = 1, 2, 3, 4, \) and the recursive equations are contained in the Appendix.

Proof. See the Appendix.

1.2 Expected inflation rates

Our model’s parameters are estimated with data that include survey forecasts of an inflation rate that begins and ends at two future dates. If the current date is \( t, \) whereas \( t + n \Delta t \) and \( t + (n + m) \Delta t \) are the dates when the inflation rate starts and ends, then the forecast of this continuously compounded inflation rate is

\[ E_t \left[ \frac{1}{m \Delta t} \ln \left( \frac{I_t + (n+m) \Delta t}{I_t + n \Delta t} \right) \right] = \frac{1}{m \Delta t} \left( E_t \left[ \ln \left( \frac{I_t + (n+m) \Delta t}{I_t} \right) \right] - E_t \left[ \ln \left( \frac{I_t + n \Delta t}{I_t} \right) \right] \right), \quad (20) \]

which is the difference between expectations of an inflation rate over two different horizons. Proposition 3 provides the formula for such an expected rate of inflation.

Proposition 3. The date \( t \) expectation of the inflation rate for a horizon of \( n \) periods is

\[ E_t \left[ \ln \left( \frac{I_{t+n \Delta t}}{I_t} \right) \right] = K^*_n + A^*_n \pi_t + B^*_n r_t + C^*_n a_t + \sum_{j=1}^{4} D^*_j h_{j,t}^2 \quad \text{for } n \geq 1, \quad (21) \]

where \( K^*_1 = 0, A^*_1 = \Delta t, B^*_1 = 0, C^*_1 = 0, D^*_{1,1} = -\frac{1}{2} \Delta t, \) and \( D^*_j = 0, \) for \( j = 2, 3, 4, \) and where the recursions are provided in the Appendix.

Proof. See the Appendix.

The Appendix shows that the market price of risk parameters, \( \phi_j, \quad j = 1, \ldots, 4, \) appear in the formulas for nominal and inflation-indexed yields (including swap rates), but are absent from the formula for a forecasted inflation rate. A benefit of combining data that reflect risk premia and data that do not is better identification of parameters that determine expectations of state variables versus those that characterize risk premia.

9 For example, \( m = 3 \) months when the forecasted inflation rate is for a future quarter of a year.
2. Data and Estimation Method

2.1 Data description

Our model is estimated with monthly data on U.S. nominal Treasury yields, survey inflation forecasts, rates of actual inflation, and inflation swap rates. Most data series are from January 1982 to May 2010, though inflation swap data only start in April 2003. Nominal Treasury yields come from two sources. First, zero-coupon yields of one, two, three, five, seven, ten, and 15 years to maturity are obtained from daily off-the-run Treasury yield curves constructed by Gurkaynak, Sack, and Wright (2007). Second, daily secondary market yields for one-, three-, and six-month Treasury bills are taken from the Federal Reserve’s H.15 release. All Treasury yields are observed at the first trading day of each month.

Survey forecasts of CPI inflation come from two sources. First, a monthly series beginning in 1982 is obtained from Blue Chip Economic Indicators (BCEI), which surveys approximately 50 economists employed by financial institutions, nonfinancial corporations, and research organizations. At the beginning of each month, participants forecast future CPI inflation for quarterly time periods, starting from the current calendar quarter and going out to at most eight quarters (two years) in the future. For January, February, and March, inflation rate forecasts for eight future quarters are made. For April, May, and June, forecasts for seven future quarters are made. For July, August, and September, forecasts for six future quarters are made, whereas for October, November, and December, forecasts for five future quarters are made. We use BCEI’s reported “consensus” forecast, which is the average of the participants’ forecasts.

Second, we use the median forecast of CPI inflation over the next ten years made by the approximately 40 participants of the Survey of Professional Forecasters (SPF). This ten-year forecast is at a quarterly frequency, starting in December 1991. Ang, Bekaert, and Wei (2007) find that SPF forecasts significantly outperform a variety of other methods for predicting inflation. Since the participants in the BCEI survey have qualifications similar to those of the SPF participants, BCEI forecasts should also possess these attractive features. Along with both sets of survey forecasts of inflation, we also constructed a monthly time series of actual CPI inflation.

In addition, we obtained bid and ask quotes of inflation swap rates for the first trading day of each month from Bloomberg for annual maturities from two to ten years, as well as for 12-, 15-, 20-, and 30-year maturities. The two- to ten-year swap maturities start in April 2003; the 12-, 15-, and 20-year inflation

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10 SPF forecasts are made at about the middle of February, May, August, and November. To align this survey with our other data, these forecasts are assumed to be at the start of the next month.

11 Survey forecasts are of seasonally adjusted CPI inflation, so our actual monthly CPI series is also seasonally adjusted. However, TIPS and zero-coupon inflation swaps are indexed to the seasonally unadjusted CPI. This difference likely has little impact on TIPS yields and swap rates, except perhaps for those of very short maturities.
swap rates start in November 2003; and the 30-year inflation swap rates start in March 2004.

Though not used to estimate our model, yields on TIPS will be compared to our model’s implied yields for inflation-indexed bonds. Zero-coupon TIPS yields were obtained from Gurkaynak, Sack, and Wright (2008), who derive them from TIPS coupon bond yields.

Table 1 shows summary statistics of our data. Panel A describes the levels and standard deviations of changes of nominal Treasury yields from 1982 to 2010. The term structure of average nominal yields is upward sloping, and the standard deviation of yield changes declines with maturity, consistent with mean reversion in short-term yields. Panel B shows similar statistics for survey inflation forecasts. Blue Chip Economic Indicators inflation forecasts averaged slightly over 3% from 1982 to 2010, whereas the SPF forecasts during the 1991-to-2010 period averaged 2.73%. The standard deviation of changes in forecasts mostly declined with maturity. The one-month forecast was volatile, which might reflect participants’ predictions of how commodity price swings would affect next month’s consumer prices.

Panel C of Table 1 gives statistics on the levels (midpoint of bid-ask quotes) and changes of inflation swap rates during the 2003-to-2010 period. The standard deviations of monthly changes generally decline with maturity, and correlations decline as the gap between maturities increase. The average levels of rates increased with maturity, consistent with a positive inflation risk premium. Recall from Equation (16) that in a frictionless market, inflation swap rates should equal the difference between equivalent-maturity, zero-coupon nominal Treasury and TIPS yields, i.e., the TIPS break-even inflation rate. Because we are unaware of any prior studies that have used inflation swap rates in term structure estimation, we detail in Figure 1 how swap rates compare to TIPS break-even rates. The top two panels in Figure 1 plot the inflation swap rates and TIPS break-even rates for five- and ten-year maturities over the April 2003 to June 2010 period.

As can be seen, the difference between the inflation swap rate and the TIPS break-even rate was fairly stable, perhaps reflecting the cost of replication, until the financial crisis. But during the crisis, this relation became distorted. The solid curve in the bottom right panel of Figure 1 shows the gap between the inflation swap rate and the TIPS break-even rate. This gap was fairly flat until the Lehman Brothers bankruptcy in September 2008, after which it increased dramatically by about 60 basis points.

What accounted for this break in historical relations? The bottom left panel of Figure 1 compares the bid-ask spread of ten-year inflation swap rates with the bid-ask spread of the ten-year TIPS, both series obtained from Bloomberg.

---

12 Fleckenstein, Longstaff, and Lustig (2010) find that Treasury supply-related factors affect the difference between inflation swap rates and TIPS break-even rates. The difference narrows when the U.S. auctions either nominal Treasuries or TIPS, but it widens when dealers have difficulties obtaining Treasury securities, such as during a period of increased repo failures.
Table 1
Summary statistics

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>0.0475</td>
<td>0.0003</td>
<td>0.1347</td>
<td>0.0243</td>
</tr>
<tr>
<td>3 months</td>
<td>0.0507</td>
<td>0.0006</td>
<td>0.1431</td>
<td>0.0134</td>
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<tr>
<td>6 months</td>
<td>0.0517</td>
<td>0.0015</td>
<td>0.1457</td>
<td>0.0122</td>
</tr>
<tr>
<td>1 year</td>
<td>0.0546</td>
<td>0.0029</td>
<td>0.1437</td>
<td>0.0120</td>
</tr>
<tr>
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<td>0.0065</td>
<td>0.1440</td>
<td>0.0125</td>
</tr>
<tr>
<td>3 years</td>
<td>0.0602</td>
<td>0.0090</td>
<td>0.1424</td>
<td>0.0126</td>
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<tr>
<td>5 years</td>
<td>0.0639</td>
<td>0.0171</td>
<td>0.1398</td>
<td>0.0123</td>
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<tr>
<td>7 years</td>
<td>0.0666</td>
<td>0.0236</td>
<td>0.1390</td>
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<td>10 years</td>
<td>0.0696</td>
<td>0.0309</td>
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<td>0.0116</td>
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<tr>
<td>15 years</td>
<td>0.0723</td>
<td>0.0349</td>
<td>0.1402</td>
<td>0.0110</td>
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Panel B: Survey inflation forecasts

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>0.0305</td>
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<td>0.0821</td>
<td>0.0331</td>
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<td>2 quarters</td>
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<td>−0.0030</td>
<td>0.0684</td>
<td>0.0095</td>
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<tr>
<td>4 quarters</td>
<td>0.0330</td>
<td>0.0150</td>
<td>0.0733</td>
<td>0.0050</td>
</tr>
<tr>
<td>6 quarters</td>
<td>0.0342</td>
<td>0.0170</td>
<td>0.0704</td>
<td>0.0043</td>
</tr>
<tr>
<td>8 quarters</td>
<td>0.0348</td>
<td>0.0190</td>
<td>0.0684</td>
<td>0.0057</td>
</tr>
<tr>
<td>10 years</td>
<td>0.0273</td>
<td>0.0223</td>
<td>0.0392</td>
<td>0.0020</td>
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</tbody>
</table>

Panel C: Zero-coupon inflation swap rates

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>Std. Dev.</th>
<th>Average</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.89</td>
<td>0.84</td>
<td>0.78</td>
<td>0.59</td>
<td>0.46</td>
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<td>0.0107</td>
<td>0.0205</td>
<td>−0.0240</td>
<td>0.0337</td>
</tr>
<tr>
<td>3</td>
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<td>0.74</td>
<td>0.63</td>
<td>0.64</td>
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<td>0.0326</td>
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</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>0.97</td>
<td>0.88</td>
<td>0.69</td>
<td>0.64</td>
<td>0.0097</td>
<td>0.0238</td>
<td>−0.0006</td>
<td>0.0331</td>
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<td></td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>0.94</td>
<td>0.77</td>
<td>0.71</td>
<td>0.0084</td>
<td>0.0250</td>
<td>0.0049</td>
<td>0.0319</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.00</td>
<td>0.87</td>
<td>0.79</td>
<td>0.0068</td>
<td>0.0264</td>
<td>0.0130</td>
<td>0.0314</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.00</td>
<td>0.92</td>
<td>0.0065</td>
<td>0.0287</td>
<td>0.0147</td>
<td>0.0331</td>
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<tr>
<td>30</td>
<td>1.00</td>
<td>0.0070</td>
<td>0.0298</td>
<td>0.0149</td>
<td>0.0343</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Treasury yields and the one month to eight quarters Blue Chip Economic Indicator survey inflation forecasts are for the period January 1982 to May 2010. The ten-year maturity survey inflation forecast is from the Survey of Professional Forecasters and is for the period December 1991 to March 2010. The zero-coupon inflation swap rates are for the period April 2003 to May 2010. The standard deviation is annualized based on monthly changes.

Since mid-2005, the inflation swap spread ranged mostly from six to ten basis points, except for a short period in September 2007 when oil prices surged and for very brief periods in 2009. In contrast, the spread on the ten-year TIPS increased from a small base of 0.5 basis points to over ten basis points during the crisis, before settling down to around four basis points. Thus, TIPS sustained a relatively larger rise in its bid-ask spread during the crisis, suggesting that it experienced a relatively large, sustained rise in illiquidity. TIPS’s illiquidity appears to explain the huge gap between the swap and TIPS break-even inflation rates. The bottom right panel in Figure 1 shows that this gap is highly correlated with TIPS’s (scaled) bid-ask spread. This evidence is consistent with Hu and Worah (2009), who attribute the spike in TIPS yields following Lehman Brothers’ bankruptcy to Lehman’s use of substantial amounts of TIPS to collateralize its repo borrowings and derivative positions. Lehman’s bankruptcy led to creditors releasing a flood of TIPS into the market.
at a time when there were few willing buyers. In contrast, how a liquidity crisis affects derivatives, such as inflation swaps, is theoretically unclear, but, as evidenced by the relatively flat swap bid-ask spread during the crisis, the effect appears minimal.  

Figure 1
Break-even inflation rates and inflation swap rates
The top two panels compare continuously compounded break-even inflation rates (zero-coupon Treasury yield minus zero-coupon TIPS yield) with continuously compounded zero-coupon inflation swap rates over the period from April 2003 to May 2010. Rates are in annual percentage terms. The top left panel is for a five-year maturity, whereas the top right panel is for a ten-year maturity. The lower left panel shows the bid-offer yield spread in basis points for the ten-year TIPS and for the ten-year inflation swap rate. The lower right panel compares the difference between the ten-year inflation swap rate and the ten-year break-even inflation rate (nominal Treasury yield minus TIPS yield) with the bid-offer spread of the ten-year TIPS scaled up by ten.

One aspect of TIPS that our analysis has ignored is the embedded put option that protects TIPS investors against deflation on the bond’s principal (but not coupon) payment. Since this option has a nonnegative value, its presence increases a TIPS’s price, and hence decreases its yield. Zero-coupon

13 Many hedge funds that had bought TIPS also were forced to sell to meet withdrawals by clients. As TIPS yields rose, break-even inflation rates fell to unreasonable levels that under normal market conditions would trigger arbitrage trades given the much higher inflation swap rates. But during the crisis, institutions abandoned relative value trades to seek safety in nominal Treasuries.

14 Inflation swap rates may have been less affected because swap dealers abandoned the hedging of their positions by trades in nominal bonds and TIPS. During the crisis, dealers may have acted merely as brokers, so that swap rates adjusted to equate the aggregate demand and supply for inflation protection, irrespective of the market prices of nominal Treasuries and TIPS. Campbell, Shiller, and Viceira (2009) conclude that during the crisis, the inflation swap market provided a more accurate assessment of inflation rates than the underlying TIPS break-even rates.
inflation swap contracts do not contain this option. Therefore, all else equal, break-even inflation based on a TIPS principal strip should be higher than the equivalent-maturity inflation swap rate. Moreover, if the financial crisis raised fears of deflation, the difference between the inflation swap rate and the TIPS break-even rate should have declined (become more negative). Yet, Figure 1 shows that exactly the opposite occurred, implying that TIPS’s rising illiquidity dominated any increase in the deflation put option that would have lowered TIPS yields.

Use of data on TIPS yields is not only problematic for the recent financial crisis. Studies by Sack and Elsasser (2004), Shen (2006), and D’Amico, Kim, and Wei (2008) reveal that the TIPS break-even inflation rate consistently fell below survey measures of inflation expectations and that TIPS yields contain a liquidity premium that, in the time period prior to 2004, was unreasonably large and difficult to explain based on any rational pricing model. Shen (2006) finds evidence of a drop in the liquidity premium on TIPS around 2004 that he attributes to the U.S. Treasury’s greater issuance of TIPS around that time, as well as to the beginning of exchange-traded funds that purchased TIPS.

This accumulated evidence on the distortions to TIPS yields led us to employ inflation swap rates and survey inflation forecasts as a more reliable reflection of real yields and expected inflation.15 In addition, by not using TIPS yields to estimate our model, we can compare our model-implied yields on inflation-indexed bonds to actual TIPS yields in order to evaluate earlier studies’ conclusions regarding the systematic mispricing of TIPS.

2.2 Estimation technique
Since our data are monthly, the model’s period is $\Delta t = 1/12$ of a year. Thus, the nominal short rate, $i_t$, is the one-month Treasury bill rate, $\pi_t$ is the rate of inflation expected over the next month, and $r_t$ is the one-month real rate. Our estimation imposes model restrictions on both the cross-section and time-series of bond yields, inflation forecasts, and inflation swap rates, which are in general assumed to be measured with the independent errors $\omega_{t,i} \sim N(0, w^2)$, $\nu_{t,i} \sim N(0, v^2)$, and $\mu_{t,i} \sim N(0, u^2)$, respectively, where the subscript $i$ denotes a given bond, inflation forecast, or swap rate maturity.

Whereas most bond yields and inflation forecasts are assumed to be observed with error, we assume perfect observation of the one-month nominal rate, $i_t = \pi_t + r_t - \phi_h h_{1,t}^2$, and the survey inflation forecast at the one-month horizon, $\pi_t$. These assumptions allow us to recover the exact one-period real rate, $r_t = i_t - \pi_t + \phi_h h_{1,t}^2$, given that $h_{1,t}$ is observed. However, to update the volatility factors $h_{i,t}, i = 1, \ldots, 4$, we also need to observe the central

15 An alternative is to use TIPS yields only during periods when they appear undistorted by large liquidity premia. Christensen, Lopez, and Rudebusch (2010) estimate a nominal and real term structure model using TIPS yields only from January 2003 to March 2008 because they state that illiquidity was high before and after this period. Another approach by Pflueger and Viceira (2011) is to estimate TIPS’s time-varying liquidity premium and remove it to create “liquidity adjusted” TIPS yields.
tendency, $\alpha_t$, which can be done if another particular bond yield is measured without error. We assume this perfectly observed yield is the five-year Treasury yield.

These assumptions allow us to observe $\pi_t$, $r_t$, and $\alpha_t$ and recover the $\epsilon_{j,t+\Delta t}$, $j = 1, \ldots, 4$, in Equations (2) and (6). In turn, this permits updating each of the volatility factors, $h_{j,t}$, $j = 1, \ldots, 4$. Given the state variables ($\pi_t$, $r_t$, $\alpha_t$, $h_{j,t}$, $j = 1, \ldots, 4$) at date $t$, all of the theoretical bond yields, inflation forecasts, and inflation swap rates can be computed. The differences between these theoretical quantities and their actual counterparts determine the measurement errors for bond yields, inflation forecasts, and inflation swap rates.

Let $n_{B1}^b, \ldots, n_{B}^b$ be the maturities of the $B$ different bonds, let $n_{F1}^f, \ldots, n_{F}^f$ be the horizons of the $F$ different inflation rate forecasts, and let $n_{S1}^s, \ldots, n_{S}^s$ be the maturities of the $S$ different swap rates. Then the month $t$ vector of observed variables is

$$Y_t = \begin{pmatrix} \ln(I_t + \Delta t / I_t) & \pi_{t+\Delta t} & r_{t+\Delta t} & \alpha_{t+\Delta t} & \epsilon_{1,t+\Delta t} & \epsilon_{2,t+\Delta t} & \epsilon_{3,t+\Delta t} & \epsilon_{4,t+\Delta t} & \omega_{1,t} & \omega_{2,t} & \omega_{3,t} & \omega_{4,t} & \Delta_{1,t} & \Delta_{2,t} & \Delta_{3,t} & \Delta_{4,t} & \mu_{1,t} & \mu_{2,t} & \mu_{3,t} & \mu_{4,t} \end{pmatrix}^\prime,$$

(22)

and can be written as a linear function of the state variables:

$$Y_t = A_t + M_t x_t + \Upsilon_t,$$

(23)

where $x_t = \left( \pi_t, r_t, \alpha_t, h_{1,t}^2, h_{2,t}^2, h_{3,t}^2, h_{4,t}^2 \right)^\prime$ is the state variable vector and $A_t$ and $M_t$ are appropriately defined vectors and matrices of the model parameters from Equations (5), (6), (18), (19), and (21). Also from these equations, the vector $\Upsilon_t$ is a function of the four stochastic drivers, $\epsilon_{j,t+\Delta t}$, $j = 1, \ldots, 4$, and the measurement errors $\omega_{1,t}, \omega_{2,t}, \omega_{3,t}, \omega_{4,t}$. Given the normally distributed $\epsilon_{j,t+\Delta t}$ and measurement errors, the parameters are estimated by maximum likelihood by recursively calculating the likelihood function based on Equation (23) for each date.

In principle, the model’s 36 parameters can be estimated in one step using Equation (23). However, the first element of $Y_t$ is the log inflation process $\ln(I_{t+\Delta t} / I_t) = \pi_t \Delta t - \frac{1}{2} \Delta t h_{1,t}^2 + \sqrt{\Delta t} \epsilon_{1,t+\Delta t}$. Using data only on $I_t$ and $\pi_t$ allows us to estimate the four parameters of the $h_{1,t}$ GARCH equation, namely, $d_{110}$ (equivalently, $\bar{h}_1$), $d_{11}, d_{12}$, and $d_{13}$. Therefore, to make overall parameter estimation more manageable, a two-step procedure is implemented in which we first estimate the four parameters of the $h_{1,t}$ process separately and the 32 other parameters are estimated in a second step using Equation (23) but with the $h_{1,t}$ process parameters fixed at those estimated in the first step.

---

16 At each monthly observation date, the bond yield maturities measured with error are the same, equal to three, six, 12, 24, 36, 84, 120, and 180 months, but because of the nature of the inflation survey data, the number of inflation forecasts, $F$, and their horizons vary over different observation months. Similarly, the number of inflation swap rates, $S$, (but not their horizons) varies over different observation months.
3. Empirical Results

3.1 Parameter estimates and state variable dynamics

Table 2 reports the first-step estimates of the parameters of the inflation volatility process, \( h_{1,t} \), using data on the CPI (\( I_t \)) and the one-month forecast of inflation (\( \pi_t \)) derived from BCEI surveys. The annualized, conditional standard deviation for inflation over a one-month horizon has a steady-state value of \( \bar{h}_1 = 88 \) basis points.\(^{17}\) The volatility of inflation displays GARCH effects since the coefficient on a shock to inflation in the GARCH updating, \( d_{12} \), is significantly positive.\(^{18}\) However, since \( d_{13} \) is insignificantly different from zero, there is no evidence that inflation’s volatility responds asymmetrically to innovations.

Table 3 reports estimates of the model’s other parameters. To gauge the statistical significance of permitting GARCH behavior, we estimated the unrestricted model as well as restricted models that assume some of the volatilities are constant, i.e., \( h_{j,t} = \bar{h}_j \). The first column of Table 3 reports estimates assuming no GARCH behavior (\( h_{j,t} = \bar{h}_j \), for \( j = 2, 3, \) and 4), whereas the second, third, and fourth columns assume GARCH behavior only for \( h_{2,t}, h_{3,t}, \) or \( h_{4,t} \), respectively. Finally, the last column of Table 3 is the unrestricted model that permits GARCH behavior for \( h_{2,t}, h_{3,t}, \) and \( h_{4,t} \).

Table 3 indicates that one can reject at the 1% level of significance the hypothesis of no GARCH behavior for each of the less restricted cases. Relative to the model with no GARCH behavior, the largest increase in

Table 2
Inflation process estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>t-Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{h}_1 )</td>
<td>0.0088</td>
<td>6.12</td>
<td>0.000</td>
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<tr>
<td>( d_{11} )</td>
<td>-2.034</td>
<td>-2.72</td>
<td>0.007</td>
</tr>
<tr>
<td>( d_{12} )</td>
<td>1.88x10^{-4}</td>
<td>3.67</td>
<td>0.000</td>
</tr>
<tr>
<td>( d_{13} )</td>
<td>2.54</td>
<td>0.159</td>
<td>0.874</td>
</tr>
</tbody>
</table>

Estimation uses monthly data on inflation and the Blue Chip Economic Indicator one-month survey forecast of inflation from January 1982 to June 2010. The total number of observations is 341. The inflation process dynamics are

\[
\frac{I_{t+\Delta t}}{I_t} = e^{\pi_t \Delta t - \frac{1}{2} h_{1,t}^2 \Delta t} + h_{1,t} \sqrt{\Delta t} \varepsilon_{1,t+\Delta t}
\]

\[
h_{1,t+\Delta t} - h_{1,t}^2 = \left[ d_{10} + d_{11} h_{1,t}^2 + d_{12} (\varepsilon_{1,t+\Delta t} - d_{13} h_{1,t}) \right] \Delta t
\]

\[
h_{1}^2 = -\frac{d_{10} + d_{12}}{d_{11} + d_{12} d_{13}^2}.
\]

\(^{17}\) Jarro and Yildirim (2003) obtain a comparable inflation volatility estimate of 87 basis points.

\(^{18}\) The process displays mean-reversion since the estimate of \( d_{11} \) is significantly different from the random walk value of \(-1/\Delta t = -12\).
Table 3  
Nominal and real term structure model estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>No GARCH</th>
<th>$h_2$ GARCH</th>
<th>$h_3$ GARCH</th>
<th>$h_4$ GARCH</th>
<th>$h_2$, $h_3$, $h_4$ GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td>0.0302***</td>
<td>0.0292***</td>
<td>0.0403***</td>
<td>0.0181***</td>
<td>0.0299***</td>
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<tr>
<td>$\omega_1$</td>
<td>0.6160***</td>
<td>0.5860***</td>
<td>0.5770***</td>
<td>0.5860***</td>
<td>0.6131***</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>-3.0914***</td>
<td>-3.0636***</td>
<td>-2.5964***</td>
<td>-3.2317***</td>
<td>-2.6702***</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>1.1578***</td>
<td>1.1240***</td>
<td>1.2272***</td>
<td>1.0610***</td>
<td>1.2591***</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-1.4722**</td>
<td>-1.4902***</td>
<td>-1.2792***</td>
<td>-1.4653***</td>
<td>-1.3370**</td>
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<tr>
<td>$\gamma_2$</td>
<td>0.0943***</td>
<td>0.0653***</td>
<td>0.1923***</td>
<td>0.2113***</td>
<td>0.2019***</td>
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Ln Likelihood | 35848 | 35891 | 36058 | 35886 | 36199 |
Reject No GARCH? | Yes | Yes | Yes | Yes | Yes |

Shown are the parameter estimates for the five models indicated in the columns. The standard deviations for the measurement errors for nominal Treasury yields, survey inflation rate forecasts, and inflation swap rates, respectively. For each set of estimates, the parameters of the GARCH process for inflation ($\phi_j$) are fixed at the point estimates reported in Table 2. Estimation uses monthly data from January 1982 to June 2010. The model dynamics are

$$\pi_{t+\Delta t} - \pi_t = \left[ a_t + a_1 r_t + a_2 \pi_t \right] \Delta t + \sqrt{\Delta t \sum_j^2 \beta_j h_{j,t}}(\epsilon_{j,t} \Delta t)$$

$$r_{t+\Delta t} - r_t = \left[ b_0 + b_1 r_t + b_2 \pi_t \right] \Delta t + \sqrt{\Delta t \sum_j^3 \gamma_j h_{j,t}}(\epsilon_{j,t} \Delta t)$$

$$a_{t+\Delta t} - a_t = \left[ \omega_0 + c_1 a_t \right] \Delta t + \sum_j^4 \rho_j h_{j,t} \epsilon_{j,t} \Delta t$$

$$h_{j,t+\Delta t} - h_{j,t} = \left[ d_{j,t} + d_1 h_{j,t}^2 + d_2 (\epsilon_{j,t} \Delta t - d_3 h_{j,t}^2) \right] \Delta t, \quad j = 1, 2, 3, 4$$

$$\bar{\pi} = -\frac{a_1 b_0 c_1 + b_1 c_0}{c_1 (a_1 b_2 - a_2 b_1)}, \quad \bar{\omega} = -\frac{a_2 b_0 c_1 + b_2 c_0}{c_1 (a_1 b_2 - a_2 b_1)}$$

$$h_{2,j}^2 = \frac{d_{j,t} + d_1 h_{j,t}^2}{d_{j,t} + d_2 d_{j,t}^2}, \quad \text{Risk premia } = \phi_j h_{j,t}, \quad j = 1, 2, 3, 4.$$
able to display GARCH, which is the independent volatility component for expected inflation, \( \pi_t \). Allowing for GARCH effects is least important for \( h_{4,t} \), the independent volatility component of the central tendency. The last column of Table 3 indicates that for the fully unrestricted case in which \( h_{2,t}, h_{3,t}, \) and \( h_{4,t} \) all follow GARCH processes, all of the GARCH volatility parameters \((d_{22}, d_{32}, \text{and } d_{42})\) are significantly positive. Based on these unrestricted model estimates and those for the inflation GARCH process in Table 2, measures of persistence for \( h_{j,t}^2, j = 1, \ldots, 4, \) can be computed. The half-life for a shock in \( h_{j,t}^2 \) to revert to its steady state of \( \tilde{h}_j^2 \) is 3.4 months, 0.7 months, 1.6 months, and 4.6 months for \( j = 1, 2, 3, \) and 4, respectively. 19

3.2 Levels of state variables

We obtain reasonable estimates for the unconditional means of inflation and the real interest rate. The unrestricted model estimate of \( \pi \) is 2.99%, just below the sample average BCEI one-month inflation forecast of 3.05% shown in Table 1. This estimate, along with the estimated steady-state one-month real rate of \( r = 1.76\% \), and the estimated steady-state inflation risk premium of \( -\phi_1 \tilde{h}_1^2 = -0.47\% \), imply from Equation (5) that the steady-state one-month nominal interest rate is \( \tilde{i} = \tilde{r} + \pi - \phi_1 \tilde{h}_1^2 = 4.28\% \), somewhat below the sample average one-month Treasury bill rate of 4.75% given in Table 1. Table 3 also shows that permitting a central tendency for inflation is important since the mean reversion parameter estimate of \( c_1 \) is \(-0.056\) and is significantly different from both zero and the no-central-tendency case of \( c_1 = -1/\Delta t = -12 \).

Figure 2 plots the model-implied state variables over the 1982-to-2010 sample period. The top panel indicates that the rate of expected inflation over one month, \( \pi_t \), trended downward since the early 1980s. At the beginning, the central tendency for inflation was above \( \pi_t \), as investors apparently thought longer-term inflation would remain high. However, the Federal Reserve may have gained credibility in lowering inflation since the central tendency later declined to the short-run expected inflation rate. Early in 2008, expected inflation rose significantly and then plunged at mid-year as the financial crisis worsened.

The bottom panel in Figure 2 displays the one-month real interest rate, \( r_t \). There was an unusually long period from mid-2002 to 2005 when it was negative, consistent with the belief that real rates were kept too low for too long and inflated a credit bubble. 20 The panel also shows that at the start of 2008, the one-month real rate was negative and then rose dramatically, consistent with the opposite movement in expected inflation as the nominal rate remained near

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19 The half-life in periods of length \( \Delta t \) (months) is \( \ln(\frac{1}{2})/\ln(1 + (d_{j1} + d_{j2}d_{j3})^2 \Delta t) \).

20 During this period, as well as the brief period in 1994 when the real rate was negative, the Federal Reserve pegged the federal funds rate below the rate recommended by the Taylor rule.
zero during this time. At the end of 2009, the real rate became negative again, consistent with the Federal Reserve’s pegging of short-term nominal rates near zero.

Our model estimates of these state variables’ processes indicate relatively strong mean-reversion for expected inflation and real rates but high persistence for inflation’s central tendency. The half-lives for the variables to return to their steady states following a deviation are 0.26, 0.65, and 12.38 years for $\pi_t$, $r_t$, and $\alpha_t$, respectively. That inflation’s central tendency displays very weak mean-reversion suggests that investors’ long-run inflation expectations are not well anchored.

### 3.3 State variable volatilities and correlations

Based on the unrestricted model’s parameter estimates, Table 4 reports statistics for the implied standard deviations and correlations for inflation, expected inflation, the real rate, and the central tendency. The first column calculates the state variables’ annualized standard deviations and correlations over a one-month horizon, assuming that each of the GARCH processes begins at their steady-state values: $h_{j,t} = \bar{h}_j$, $j = 1, \ldots, 4$. The real interest rate, $r_t$, and
expected inflation, $\pi_t$, have the highest unconditional standard deviations of 3.26% and 3.14%, respectively. Conditional on its mean of $\pi_t$, the steady-state one-month standard deviation of log inflation is 0.88%, whereas the steady-state standard deviation of the central tendency is 1.09%. One also sees that an innovation in actual inflation ($I_t + \Delta t$) has a 0.35 correlation with an innovation in expected inflation ($\pi_t + \Delta t$) and a 0.13 correlation with an innovation in the central tendency ($\alpha_t$). This suggests that when investors experience a positive inflation surprise, their one-month expectation of inflation is partially updated and, to a lesser degree, so is their longer-horizon expectation of inflation via the central tendency.

We also see that at the steady state, the one-month expected inflation and real rate are strongly negatively correlated at $-0.87$. This is consistent with Benninga and Protopapadakis (1983), Summers (1983), and Pennacchi (1991) and is likely a consequence of Federal Reserve policy that keeps short-term nominal rates stable by pegging the federal funds rate. Controlling the short-run nominal rate implies that any change in short-run inflation expectations must lead to an offsetting change in the short-run real rate. Evidence by Ang, Bekaert, and Wei (2007) confirms that the short-term real rate is quite variable.

Of course, because of GARCH behavior, the state variables’ standard deviations and correlations are not constant. Columns 2, 3, and 4 of Table 4 calculate the model-implied average, minimum, and maximum of the standard deviations and correlations over the sample period. From the minimum and maximum values, we see that standard deviations and correlations varied significantly. The central tendency’s correlation with real rates and expected inflation even changed signs. The standard deviations of expected inflation and the real rate were especially high during the early 1980s, when the Federal Reserve was battling to lower inflation expectations, and also during the late 2000s, when commodity price volatility picked up and the financial crisis hit.

### Table 4

<table>
<thead>
<tr>
<th>Time variation of standard deviations and correlations</th>
<th>1982 to 2010 Sample Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Average</td>
</tr>
<tr>
<td>ln($I_t + \Delta t$)</td>
<td>0.0088</td>
</tr>
<tr>
<td>$\pi_t + \Delta t$</td>
<td>0.0314</td>
</tr>
<tr>
<td>$r_t + \Delta t$</td>
<td>0.0326</td>
</tr>
<tr>
<td>$\alpha_t + \Delta t$</td>
<td>0.0109</td>
</tr>
</tbody>
</table>

The annualized standard deviations and correlations are for one-month horizons and based on parameter estimates from the unrestricted model. The data are monthly over the period January 1982 to June 2010.
3.4 The model’s fit to the data

3.4.1 Nominal yields. Over our 1982-to-2010 sample period, the average nominal yield measurement errors (difference between the observed data yields and the model-implied yields) across all maturities is less than three basis points, with the largest errors occurring early in the period during a time of extreme interest rate volatility. Indeed, the mean error from 1990 to 2010 is less than one basis point. As reported in the last column of Table 3, the estimated standard deviation of measurement errors for nominal yields is $w = 36$ basis points, close to the average standard deviation of errors across all maturities over the sample period of 33.5 basis points. The ten-year maturity has a standard deviation of 27 basis points. The largest standard deviation of errors is for the three-month rate (42 basis points).

Compared to other studies that fit both nominal and real term structures, our results are satisfactory. For example, Chen, Liu, and Cheng (2010) estimate a multifactor Cox, Ingersoll, and Ross (1985) model using both nominal Treasury and TIPS data and for nominal yields obtain an average measurement error of 24 basis points and an average measurement error standard deviation of 74 basis points. Christensen, Lopez, and Rudebusch (2010) fit a multifactor Gaussian model to nominal and TIPS yields, finding measurement errors for ten-year nominal yields to average ten basis points and have a standard deviation of 11 basis points.

3.4.2 Inflation swap rates. For an inflation swap of a given maturity, the average measurement error over the sample is close to zero, with the possible exception of the 15-year inflation swap, which has a bias of 11 basis points. The sample standard deviation of errors for a given maturity is typically close to the $u = 28$ basis point estimate reported in the last column of Table 3.

Examining the average measurement errors across maturities at each monthly date during our sample, the errors stay within a band of 50 basis points from zero, except during November 2008, when the errors exceed 100 basis points. Whereas during the financial crisis our model over predicted actual inflation swap rates, recall from Figure 1 that TIPS break-even inflation rates were even smaller than were swap rates at that time. Thus, had we used TIPS in our model estimation, measurement errors would likely have been much larger. Indeed, in fitting their model to TIPS, Christensen, Lopez, and Rudebusch (2010) obtained huge measurement errors during this period.

3.4.3 Survey forecasts of inflation. Examining the difference between actual and model-implied survey forecasts of inflation, on average the model over predicts two- and three-quarter BCEI inflation forecasts by less than seven basis points but under predicts seven- and eight-quarter inflation forecasts by around six and nine basis points, respectively. The bias for the ten-year SPF forecast is less than one basis point. The sample standard deviations
of measurement errors across maturities range from 48 to 36 basis points, consistent with the \( \nu = 40 \) basis point standard deviation estimate reported in the last column of Table 3.

By calculating the average measurement error across all forecast maturities for each monthly date, we find that the model tended to under predict survey forecasts during the late 1980s and over predict during the late 1990s. During most of the recent financial crisis, the model under predicted expected inflation relative to the survey forecasts. Recall that during this same time the model was over predicting inflation swap rates. Hence, during the crisis the model was making a compromise between the relatively high survey inflation expectations and the relatively low inflation expectations reflected in swap rates.

3.5 Nominal bond yields, risk premia, and volatility

This section investigates the model-implied term structure of nominal yields, nominal bonds’ risk premia, and the volatility of nominal yields. Figure 3 shows the model’s nominal yield curve (solid line) when all variables equal their steady states. Yields appear reasonable, even for maturities out to 30 years, a horizon in which no Treasury data were used. The slopes of this steady-state nominal yield curve (difference between yields and the steady-state one-month nominal rate of \( i = 4.28\% \)) equal 114, 177, 236, and 257 basis points at the five-, ten-, 20-, and 30-year maturities, respectively. Moving from

![Figure 3](http://rfs.oxfordjournals.org/)

**Figure 3**

Steady-state yield curves and expected excess returns

The graph shows the nominal and inflation-indexed (real) yield curves when all state variables are at their steady-states. Also shown are the expected excess nominal returns on nominal bonds and the expected excess real returns on real bonds (expected returns relative to one-month maturity return).
this steady-state yield curve, let us now consider the time-series properties of yields.

Time-series Properties: Campbell-Shiller Tests and Bond Risk Premia

Note that rates of return on n-period bonds are given by

\[ r_{j,t+\Delta t}^{(n)} = \ln \left( \frac{P_{j,t+\Delta t}^{(n-1)}}{P_{j,t}^{(n)}} \right) = \gamma_{j,t}^{(n)} (n\Delta t) \]

\[ - y_{j,t+\Delta t}^{(n-1)} (n - 1)\Delta t, \quad \text{for } j = N, R, \]

(24)

where if \( j = N (j = R) \), Equation (24) denotes the nominal (real) rate of return on a nominal (real) bond. Since \( y_{N,t}^{(1)} = i_t \) and \( y_{R,t}^{(1)} = r_t \), Equation (24) implies that the corresponding expected excess returns (risk premia) equal

\[ \pi_{j,t}^{(n)} = E_t \left[ r_{j,t+\Delta t}^{(n)} \right] - y_{j,t}^{(1)} \], \quad \text{for } j = N, R

\[ = \left( \gamma_{j,t}^{(n)} - E_t \left[ y_{j,t+\Delta t}^{(n-1)} \right] \right) (n - 1) + s_{j,t}^{(n)}, \]

(25)

where \( s_{j,t}^{(n)} = y_{j,t}^{(n)} - y_{j,t}^{(1)} \) is the slope of the yield curve. The Appendix shows that these return risk premia, \( \pi_{j,t}^{(n)} \), for our model are linear functions only of the four volatility-state variables, \( h_{j,t}^2 \), \( j = 1, \ldots, 4 \). Figure 3 shows that if these state variables equal their steady states (\( h_{j,t}^2 = \bar{h}_j^2 \)), then both nominal and real risk premia are concave functions of maturity (dotted and short-dashed lines). Many empirical studies, notably Fama and Bliss (1987) and Campbell and Shiller (1991), provide overwhelming evidence of significant time variation in nominal bond risk premia. We now consider whether our model fits the patterns documented by prior empirical research. Equation (25) can be rearranged as

\[ E_t(y_{j,t+\Delta t}^{(n-1)}) - y_{j,t}^{(n)} = \frac{s_{j,t}^{(n)}}{n - 1} - \frac{\pi_{j,t}^{(n)}}{n - 1}, \quad j = N, R. \]

(26)

Based on this equation, consider the following regression:

\[ y_{j,t+\Delta t}^{(n-1)} - y_{j,t}^{(n)} = \beta_{j,0}^{(n)} + \beta_{j,1}^{(n)} \frac{s_{j,t}^{(n)}}{n - 1} + \beta_{j,2}^{(n)} \frac{\pi_{j,t}^{(n)}}{n - 1} + \epsilon_{j,t+\Delta t}^{(n)}, \quad j = N, R. \]

(27)

For nominal yields \( j = N \), Campbell and Shiller (1991) examined the “Expectations Hypothesis” by setting \( \beta_{N,2}^{(n)} = 0 \) and testing if \( \beta_{N,1}^{(n)} = 1 \). Their tests rejected this hypothesis, and estimates for \( \beta_{N,1}^{(n)} \) became increasingly negative as maturity, \( n \), increased. Allowing \( \beta_{N,2}^{(n)} \) to be unconstrained and using a three-factor Gaussian model with the “essentially affine” risk premium structure of Duffee (2002), empirical tests by Dai and Singleton (2000) could not reject the hypothesis that \( \beta_{N,1}^{(n)} = 1 \) and \( \beta_{N,2}^{(n)} = -1 \).
We repeat the Campbell-Shiller regressions \((\beta_{N,2}^{(m)})\) constrained to zero) using actual monthly nominal Treasury yields for our 1982-to-2010 sample period, and estimates of the slope coefficient \(\beta_{N,1}^{(m)}\) are shown in the first column of Table 5. The null hypothesis that \(\beta_{N,1}^{(m)} = 1\) is rejected for all maturities at

### Table 5

**Campbell-Shiller regressions for nominal yields**

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<tr>
<th>Maturity (Years)</th>
<th>Actual Expectations</th>
<th>Including Model Risk Premium</th>
<th>Joint Test</th>
<th>Model-Implied Expectations</th>
<th>Including Model Risk Premium</th>
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</tbody>
</table>

The left panel shows the results for Campbell-Shiller regressions of the monthly changes in actual nominal yields against the adjusted slope (slope divided by maturity in months minus 1) over the period January 1982 to May 2010:

\[
y_{N,t}^{(m-1)} - y_{N,t}^{(m)} = \beta_{N,0}^{(m)} + \beta_{N,1}^{(m)} \frac{y_{N,t}^{(m)}}{m-1} + \epsilon_{N,t}^{(m)}.
\]

For each maturity (in years), the coefficients of the adjusted slope are shown together with the standard error and the \(t\)-statistic for the null hypothesis that \(\beta_{N,1}^{(m)} = 1\). The column next to the slope column reports the \(p\)-value for the test \(\beta_{N,1}^{(m)} = 1\). The next two columns report the results for regressions that include the model-implied time varying risk premium:

\[
y_{N,t+\Delta t}^{(m-1)} - y_{N,t}^{(m)} = \beta_{N,0}^{(m)} + \beta_{N,1}^{(m)} \frac{y_{N,t}^{(m)}}{m-1} + \beta_{N,2}^{(m)} \frac{\pi_{N,t}^{(m)}}{m-1} + \epsilon_{N,t+\Delta t}^{(m)}.
\]

For each maturity, the coefficients of the adjusted slope are shown together with the standard error and the \(t\)-statistic for the null hypothesis that \(\beta_{N,1}^{(m)} = 1\). The coefficients of the adjusted risk premium are shown together with the standard error and the \(t\)-statistic for the null hypothesis that \(\beta_{N,2}^{(m)} = -1\). The \(p\)-value is now for the joint test that \(\beta_{N,1}^{(m)} = 1\) and \(\beta_{N,2}^{(m)} = -1\). The right panel repeats the tests using the model-implied nominal yields.
the 10% significance level, and estimates become more negative as maturity increases. The second column of the table reports results of regression (27) with our model-implied risk premia, \( \pi_{N,t}^{(n)} \), computed at each date. There, one sees that the joint hypothesis that the slope is 1 and the excess return coefficient is \(-1\) cannot be rejected at the 10% significance level, except for the ten- and 15-year maturities. In these regressions, the dependent variable and slope are from actual Treasury yields. Replacing them by model-implied yields for each date from 1982 to 2010, similar results are obtained, as shown in the right-most panel of Table 5.

Thus, our model generates time-varying risk premia that are, for most maturities, not inconsistent with the theoretical relation (26). The variation can be substantial. Excess expected returns on longer-dated bonds became negative several times, almost always during episodes of financial market turmoil: May to December 1982 (Paul Volker’s Federal Reserve sharp tightening of monetary policy); November to December 1987 (stock market crash); June to August 1997 (Asian financial crisis); October 1998 (Russian financial crisis); June to July 2000 (bursting of dot.com bubble); October and November 2001 (September 11 terrorist attack); November 2007 to April 2008 (worsening of the subprime crisis); and October 2008 to April 2009 (period following Lehman Brothers’ bankruptcy). That the model predicts negative risk premia during financial crises may be reassuring. There is consensus that investors view Treasury bonds as a “safe haven” during a crisis and bid up their prices, thereby reducing excess expected returns.

Whereas our model’s risk premia are entirely determined by the volatility factors \( (h_{j,t}, j = 1, \ldots, 4) \), Figure 4 shows that they play a minor role in determining the cross-section of yields. The figure illustrates the contribution of each of the seven state variables to the six-month, two-year, five-year, and ten-year yields. Since the four volatility-state variables add minimally to the yields, their collective contribution is shown. Nominal yields are largely determined by expected inflation, \( \pi_t \), the real rate, \( r_t \), and the central tendency, \( \alpha_t \), with the importance of the last state variable increasing as maturity increases. Only occasionally do the volatility-state variables play a significant role, and then only for shorter maturities. Our model’s results align with Duffee (2011), who constructs a multifactor Gaussian model in which a subset of factors describe bond yields and a mutually exclusive subset of “hidden factors” determine bond risk premia.

**Volatility Effects**

Unlike Gaussian models, our model also captures the empirical property that yields have stochastic volatilities that are weakly linked to yield levels. The Appendix shows that the covariance of yields are linear functions solely of the four volatility-state variables, \( h_{j,t}^2, j = 1, \ldots, 4 \). When these volatility variables are at their steady states, the volatility of nominal yields (annualized
Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps

Figure 4
Decomposition of nominal yields
The panels show the contributions of the one-month expected inflation ($\pi_t$), the one-month real rate ($r_t$), the central tendency of inflation ($\alpha_t$), and the volatility-state variables ($h_{jt}, j = 1, \ldots, 4$) to the six-month, two-year, five-year, and ten-year maturity nominal yields.

standard deviation of monthly changes) is at a minimum at the two-year maturity of just under 1%, rises to a maximum of 1.2% at the eight-year maturity, and then falls again to less than 1% at the 20-year maturity. The top left panel of Figure 5 displays the term structures of yield volatilities over our sample, indicating significant variation at the short end, during the early 1980s and early 1990s. The bottom left panel is a scatterplot of the volatility of the five-year maturity nominal yield against the yield’s level. It indicates a positive relation, with the slope of the regression line being different from zero at the 1% significance level.

Also in contrast to Gaussian models, our model’s yield changes display excess kurtosis and skewness, with changes in yields of one-year maturity and less tending to display negative skewness, whereas longer-maturity yield changes are positively skewed. Correlations between yields are also time-varying. For example, over our sample period the model-implied correlation between one-month changes in the one- and ten-year yields averages 0.37 but reaches a maximum and minimum of 0.67 and $-0.05$, respectively.

Collectively, then, whereas completely affine, our model captures the time-series properties of nominal yields fairly well. Given its reasonable

21 For example, average sample skewness for a one-year change in the three-month and ten-year yields is $-0.43$ and 0.46, respectively.
performance for describing nominal yields, we next consider their decomposition into real and inflation components.

### 3.6 Real bond yields, risk premia, and volatility

Figure 3 shows the model-implied inflation-indexed (real) yield curve when all variables equal their steady states (long dashed line). The slopes of this real yield curve (difference between yields and the steady-state one-month real rate of $\bar{r} = 1.76\%$) equal 52, 96, 149, and 177 basis points at the five-, ten-, 20-, and 30-year maturities, respectively. Similar to the steady-state nominal yield curve, this real yield curve is concave and upward sloping but less steep. Panel A of Figure 6 shows much time-series variation in the term structure of real yields over the 1982-to-2010 sample period, particularly at short maturities.

Since Equation (26) holds both in real and nominal terms, regression (27) relating yields to term structure slopes and risk premia should also hold for real yields with $\beta_{R,1}^{(n)} = 1$ and $\beta_{R,2}^{(n)} = -1$. Being unaware of prior research examining regression (27) for real rates, we carried out such a test of our model over two different sample periods. First, during 2003 to 2010 when inflation swap data are available, Equation (16) was used to obtain “actual” zero-coupon...
real yields, $y_{R,t}^{(n)}$, by subtracting the inflation swap rate from the equivalent maturity nominal Treasury yield. From these real yields, slope variables, $s_{R,t}^{(n)}$, were calculated. Second, over the entire 1982-to-2010 period, we calculated
model-implied slopes $s_{R,t}^{(n)}$ from the model-implied real yields (as shown in Panel A of Figure 6). For both the first and second samples, the model-implied real risk premia, $\pi_{R,t}^{(n)}$, were calculated from the time series of the volatility-state variables.

Table 6 reports results of regression (27) both with the risk premium coefficient $\beta_{R,2}^{(n)}$ restricted to zero (Expectations Hypothesis) and unrestricted.

Table 6 Campbell-Shiller regressions for real yields

<table>
<thead>
<tr>
<th>Maturity (Years)</th>
<th>Actual</th>
<th>Model-Implicit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expectations Hypothesis</td>
<td>Including Model Risk Premium</td>
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<tr>
<td></td>
<td>Slope</td>
<td>Joint Test $p$-Value</td>
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<tr>
<td>2</td>
<td>0.21</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>(0.77)</td>
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<tr>
<td></td>
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<td></td>
<td>0.58</td>
<td></td>
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<td></td>
<td>(0.88)</td>
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<tr>
<td>3</td>
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<tr>
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<td>10</td>
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<td>15</td>
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</tr>
<tr>
<td></td>
<td>(2.48)</td>
<td></td>
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<tr>
<td></td>
<td>0.51</td>
<td></td>
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</tbody>
</table>

The left panel shows the results for Campbell–Shiller regressions of the monthly changes in actual real yields against the adjusted slope (slope divided by maturity in months minus 1) over the period January 2003 to May 2010:

$$\gamma_{R,t+\Delta t}^{(n-1)} - \gamma_{R,t}^{(n)} = \beta_{R,0}^{(n)} \gamma_{R,t}^{(n)} + \beta_{R,1}^{(n)} \gamma_{R,t}^{(n)} + \gamma_{R,t+\Delta t}^{(n)}.$$

Real yields in this panel were derived as the difference between the equivalent maturity nominal Treasury yield and the inflation swap rate. For each maturity, the coefficients of the adjusted slope are shown together with the $t$-statistic for the null hypothesis that $\beta_{R,1}^{(n)} = 1$. The column next to the slope column reports the $p$-value for the test $\beta_{R,1}^{(n)} = 1$.

The next two columns report the results for regressions that include the model-implied time-varying risk premium:

$$\gamma_{R,t+\Delta t}^{(n-1)} - \gamma_{R,t}^{(n)} = \beta_{R,0}^{(n)} \gamma_{R,t}^{(n)} + \beta_{R,1}^{(n)} \gamma_{R,t}^{(n)} + \beta_{R,2}^{(n)} \gamma_{R,t+\Delta t}^{(n)}.$$

For each maturity, the coefficients of the adjusted slope are shown together with the standard error and the $t$-statistic for the null hypothesis that $\beta_{R,1}^{(n)} = 1$. The coefficients of the adjusted risk premium are shown together with the standard error and the $t$-statistic for the null hypothesis that $\beta_{R,2}^{(n)} = -1$. The $p$-value is now for the joint test that $\beta_{R,1}^{(n)} = 1$ and $\beta_{R,2}^{(n)} = -1$. The right panel repeats the tests using the model-implied real yields.
Interestingly, there are qualitative differences compared to the nominal results of Table 5. Using the “actual” real rates from 2003 to 2010, the left panel reports that the regression, including only the slope term, does not reject the hypothesis $\beta_{R,1}^{(n)} = 1$. Unlike the nominal results, the estimates $\beta_{R,1}^{(n)}$ become more positive as maturity increases. When the real risk premium, $\pi_{R,t}^{(n)}$, is included in the regression, rejection of the joint hypothesis of $\beta_{R,1}^{(n)} = 1$ and $\beta_{R,2}^{(n)} = -1$ at a 10% level of significance occurs only for the ten-year maturity, but this happens because $\beta_{R,1}^{(n)}$ becomes too positive, whereas $\beta_{R,2}^{(n)}$ becomes too negative, just the opposite of the nominal regressions where rejection occurred because $\beta_{R,1}^{(n)}$ became negative, whereas $\beta_{R,2}^{(n)}$ became positive.

These findings could be partly due to the short 2003-to-2010 sample. When in the right panel of Table 6 we use the model-implied real yields from 1982 to 2010, the estimate of $\beta_{R,1}^{(n)}$ does decline when the slope alone is included in the regression. However, except for the 15-year maturity, the estimated slope coefficient is positive, and the hypothesis that $\beta_{R,1}^{(n)} = 1$ cannot be rejected. Notably, when the real risk premium, $\pi_{R,t}^{(n)}$, is included, the model fits well at the longer maturities but is rejected at the shorter ones. Again, however, rejection at short maturities occurs because $\beta_{R,1}^{(n)}$ exceeds 1, whereas $\beta_{R,2}^{(n)}$ is lower than $-1$.

**Volatility Effects**

It is apparent from the top panel of Figure 6 that short-maturity real yields tend to be more volatile than longer-maturity ones, consistent with the top right panel of Figure 5 that shows the term structures of real yield volatilities over our sample period. The bottom right panel in Figure 5 displays a scatter diagram of the levels of five-year real yields versus their volatilities over our sample period and indicates, like nominal yields, a positive relation between yields’ levels and their volatility. The slope of the regression line is statistically different from zero at the 1% level of significance.

The high volatility of short-term real yields is consistent with Figure 2’s reported high standard deviation of the one-month real rate, $r_t$. Since the Federal Reserve’s policy pegs short-term nominal rates, changes in short-run inflation expectations induce almost opposite changes in short-run real yields.

**3.7 Expected inflation and inflation and real risk premia**

Panel B of Figure 6 shows our model’s implied term structures of expected inflation for each month from 1982 to 2010. Consistent with the evidence in Figure 2 of a falling central tendency, inflation expectations trended downward

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22 When $h_{j,t} = h_{j, t-1, \ldots, 4}$, the real yield volatilities (annualized standard deviation of monthly changes) are 112 basis points at the one-year maturity, falling to 68 basis points at the four-year maturity, rising to 72 basis points at the eight-year maturity, and gradually falling to 61 basis points at the 20-year maturity.
at all maturities. However, the term structure was often upward sloping during the mid-1980s, suggesting that investors were not yet convinced that inflation would remain low in the longer run. Expected inflation can be volatile at short maturities but is smoother at longer horizons.

Other components of nominal yields that interest policymakers and academicians are the term structures of inflation and real risk premia. Saving the cost of an inflation risk premium is used to justify a government’s issuance of inflation-indexed bonds, and this premium must first be quantified to derive inflation expectations from an inflation swap or TIPS break-even inflation rate. In addition, the term structure of real risk premia provides information on how the real cost of financing varies by maturity.

We quantify the term structures of inflation and real risk premia in the following manner.\(^\text{23}\) Consider our model’s nominal and real yield curves when all of the market prices of risk are set to zero, i.e., \(\phi \equiv (\phi_1 \phi_2 \phi_3 \phi_4) = 0\). Since yields on nominal and inflation-indexed bonds maturing in \(n\) periods are denoted as \(y^{(n)}_{j,t}\), \(j = N, R\), respectively, let their zero-risk premium counterparts be \(y^{(n)}_{j,t} (\phi = 0)\), \(j = N, R\). Then denote the date \(t\), \(n\)-period real and inflation risk premia as \(\Phi^{(n)}_{R,t}\) and \(\Phi^{(n)}_{\text{inf},t}\), respectively, with their sum defined as the nominal risk premium, \(\Phi^{(n)}_{N,t}\). These premia are computed as

\[\Phi^{(n)}_{j,t} \equiv y^{(n)}_{j,t} - y^{(n)}_{j,t} (\phi = 0), \quad j = N, R,\]

\[\Phi^{(n)}_{\text{inf},t} \equiv \left( y^{(n)}_{N,t} - y^{(n)}_{R,t} \right) - \left( y^{(n)}_{N,t} (\phi = 0) - y^{(n)}_{R,t} (\phi = 0) \right).\]

The logic behind definition (29) is that the break-even inflation rate in parentheses includes expected inflation, the inflation risk premium, and convexity terms unrelated to risk premia. By subtracting the zero-risk premium break-even inflation rate in square brackets, expected inflation and convexity terms are eliminated, leaving only the inflation risk premium. From inspection of the formulas for nominal and real yields, given in the Appendix, it can be seen that all of these risk premia are affine functions of only the four volatility-state variables \(h^2_{j,t}\), \(j = 1, \ldots, 4\).

The term structures of nominal, real, and inflation risk premia when the volatility-state variables are at their steady states are plotted in Figure 7. The real risk premia equal 54, 102, 170, and 214 basis points at the five-, ten-, 20-, and 30-year maturities, respectively. The inflation risk premia equal 17, 45, 80,

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\(^{23}\) Christensen, Lopez, and Rudebusch (2010) define the inflation risk premium as \(\gamma^{(n)}_{N,t} - \gamma^{(n)}_{R,t} - \frac{1}{\Delta t} \ln \left( E \left[ I_{t+n\Delta t} / I_t \right] \right)\), and Chernov and Mueller (forthcoming) and Hordahl and Tristani (2007) define it as \(\gamma^{(n)}_{N,t} - \gamma^{(n)}_{R,t} - E \left[ \frac{1}{\Delta t} \ln \left( I_{t+n\Delta t} / I_t \right) \right]\). These definitions neglect convexity effects, and the former one also introduces a downward bias that increases with maturity since by Jensen’s Inequality, \(\frac{1}{\Delta t} \ln \left( E \left[ I_{t+n\Delta t} / I_t \right] \right) \geq E \left[ \frac{1}{\Delta t} \ln \left( I_{t+n\Delta t} / I_t \right) \right]\). Our approach accounts for convexity effects, is the same as Chen, Liu, and Cheng (2010), and is quantitatively similar to the latter definition for maturities out to ten years.
Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps

Figure 7: Real and inflation risk premia for steady state
The figure shows the term structures of the nominal, real, and inflation risk premia when all volatility-state variables are at their steady states. The nominal risk premium is the sum of the real and inflation risk premia.

and 100 basis points at the five-, ten-, 20-, and 30-year maturities, respectively. These steady-state term structures are very close to the average of the model estimates for each month over our sample period.

Note that the steady-state inflation risk premia are negative for short maturities, reaching a minimum of −20 basis points at a seven-month maturity before becoming positive at a 29-month maturity. An explanation is that our model is capturing the relative attractiveness of shorter maturity Treasuries, particularly Treasury bills, because of their high liquidity or “moneyness.” This is plausible since our longer-maturity yield data were constructed by Gurkaynak, Sack, and Wright (2007) from off-the-run Treasury notes and bonds, whereas our shorter-maturity data are actual yields of Treasury bills that trade in more liquid money markets. A negative inflation premium is the model’s way of adjusting for the greater liquidity at the short end of the nominal yield curve.24

24 Other research recognizes that bill yields are reduced by high liquidity. Campbell, Shiller, and Viceira (2009) add a parameter to their model estimation to “capture the liquidity effects which lower the yields on Treasury bills relative to the longer-term real yield curve.” Also, note that Figure 7 indicates a negative real risk premia at very short maturities, which is due to our real yield curve truly being yields for inflation-indexed bonds having
We next analyze the time variation of inflation and real risk premia and the volatility factors that drive their movements. Similar to Figure 4 that displays the components of nominal yields, Figure 8 shows the components of real and inflation risk premia for maturities of two, five, and ten years. The left-most panels show the time series of real risk premia which have five components: a constant and contributions from each of the four volatility-state variables. As might be expected, the main determinant of the two-year real risk premium is \( h_{3,t} \), the volatility factor determined by shocks to the one-month real interest rate, \( r_t \). However, for the five- and ten-year maturity real risk premia, the dominating volatility factor becomes \( h_{4,t} \), which is determined by shocks to inflation’s central tendency, \( \alpha_t \). This finding—that long-maturity real risk premia increase with longer-run inflation uncertainty—is consistent with research, such as Lucas (1973), Fischer (1981), Cuikerman (1982), and Fischer (1991), showing that inflation uncertainty can cause resource misallocation that slows real economic growth. Wright (2011) finds that term premia on nominal government bonds declined more in countries that reduced inflation uncertainty. Our results suggest that the decline may be due to a reduction in both real and inflation risk premia.

The right-most panels of Figure 8 show the time series of two-, five-, and ten-year inflation risk premia and the contributions of the volatility factors. For none of these inflation risk premia does the \( h_{3,t} \) volatility derived from the independent shock to the short-term real rate, \( r_t \), play a significant role. As is sensible, the volatility factor determined by shocks to short-term unanticipated inflation, \( h_{1,t} \), is the primary driver of changes in the two-year inflation risk premium, with a rise in \( h_{1,t} \) reducing the inflation risk premium. Increases in \( h_{2,t} \) and \( h_{4,t} \), which are the volatility factors determined by shocks to one-month expected inflation, \( \pi_t \), and inflation’s central tendency, \( \alpha_t \), respectively, play lesser roles in raising the two-year inflation risk premium. It is also intuitive that, as maturities increase to the five- and ten-year horizons, the role of \( h_{4,t} \) derived from shocks to inflation’s longer-run central tendency becomes dominant. Uncertainty regarding inflation’s central tendency may proxy for the credibility of monetary policy, suggesting that the longer-run inflation risk premia are linked to beliefs of central bank behavior. There are, however, a few striking instances when the \( h_{1,t} \) factor derived from shocks to unanticipated inflation plays a big role by reducing the inflation risk premium, even for long maturities. Figure 8 shows that these instances often coincide with financial

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\( \phi_1 \)

This negative relation between \( h_{1,t} \) and the inflation risk premium is consistent with our positive estimate of \( \phi_1 \) and the one-month nominal rate relation

\[
i_t = r_t + \pi_t - \phi_1 h_{1,t}^2.
\]
stress so that a “flight-to-quality” attraction for Treasuries is reflected as a decline in the inflation risk premium.26

Our average estimates for inflation risk premia are in the middle range of those found by other studies using different data and different sample periods.27 However, our estimates’ ranges of variation may be smaller than most: Figure 8 shows that the two-, five-, and ten-year inflation risk premia display basis point ranges of $-80$ to $16$, $-16$ to $30$, and $25$ to $59$, respectively. Some studies may find more volatile inflation risk premia because they use

26 Times when the five-year inflation risk premium turned negative include November 2007 to January 2008 (subprime crisis) and October 2008 to February 2009 (Lehman bankruptcy). Earlier we noted that the “safe haven” quality of nominal Treasuries reduced their excess return during periods of stress. Thus, this reduction in return typically takes the form of a declining inflation risk premium.

27 Table 1 in Chen, Liu, and Cheng (2010) summarizes empirical studies estimating inflation risk premia.
data on TIPS yields, that arguably incorporate a volatile liquidity premium, or realized macroeconomic quantities that can introduce excess “noise” in model-implied yields. Kim (2009) argues that state variables, such as realized inflation or realized output growth, are unlikely to be spanned by bond returns and realistically fail to satisfy dynamic term structure models’ no-arbitrage assumptions. As a result, model-implied yields that assume such macro-state variables may inherit unspanned noise.28

3.8 Comparison to TIPS yields
In the spirit of an out-of-sample test, we relate our model’s implied yields for inflation-indexed bonds to the actual yields of TIPS.29 Figure 9 compares five- and ten-year zero-coupon TIPS yields from Gurkaynak, Sack, and Wright (2008) for the period January 1999 to June 2010 to our model’s implied five- and ten-year zero-coupon yields for inflation-indexed bonds. Our model’s yields are significantly below TIPS yields from 1999 to 2004 and are very close to TIPS yields from 2004 to mid-2008. Starting in mid-2008, the model-implied yields again fall below the yields on TIPS until they converge in late 2009. Thus, at the beginning of our sample and during the height of the financial crisis, our model overprices inflation-indexed bonds relative to TIPS. One interpretation of this comparison is that our model performs poorly in valuing inflation-indexed bonds, except during 2004 to mid-2008 and again starting at the end of 2009.

However, recall that prior studies, such as Sack and Elsasser (2004), Shen (2006), and D’Amico, Kim, and Wei (2008), conclude that TIPS were significantly undervalued prior to 2004, and more recently, Campbell, Shiller, and Viceira (2009), Hu and Worah (2009), and Christensen, Lopez, and Rudebusch (2010) argue that there was panic selling of TIPS in the latter half of 2008 that drove their yields above, and TIPS break-even inflation below, reasonable levels. Therefore, a more logical interpretation is that the difference in the two curves in Figure 9 is confirming an erratic liquidity premium in TIPS. The model’s close fit to TIPS beginning in 2004 might be partly attributed to the initiation of a U.S. inflation swap market in 2003 that in normal times allowed dealers to arbitrage significant underpricing of TIPS (Fleckenstein, Longstaff, and Lustig 2010). In summary, the overall evidence

28 For example, two studies that find more volatile inflation risk premia are Chernov and Mueller (forthcoming), who use quarterly realized inflation and linearly de-trended per capita GDP growth as macro-state variables, and Hordahl and Tristani (2007), who use monthly realized inflation and deviations of industrial production from a quadratic trend as macro-state variables. Hordahl and Tristani (2007) also use euro-denominated inflation-indexed bonds and mention that the negative inflation risk premia that they find during 2001 to 2002 may be due to the bonds’ higher illiquidity during that period. Similarly, Christensen, Lopez, and Rudebusch (2010) state that their estimated inflation risk premium may be affected by TIPS’s liquidity premium.

29 Fleckenstein, Longstaff, and Lustig (2010) compare TIPS to a synthetic real bond constructed from nominal Treasuries and inflation swaps. They find, like us, that TIPS were often underpriced. However, their analysis can start only in July 2004, when inflation swap data became available. Using our model-implied real yields, our analysis can begin earlier in January 1999, when TIPS became available.
Figure 9
TIPS yields versus model-implied inflation-indexed yields
The figures compare the time series of TIPS yields to the model-implied inflation indexed yields from 1999 to 2010. The top panel shows the five-year maturity, whereas the bottom panel displays the ten-year maturity.

supports our earlier arguments for using data on inflation swaps and survey forecasts of inflation, rather than TIPS, to estimate a nominal and real term structure model.

4. Conclusion
From June 2004 to June 2005, the Federal Reserve raised the target federal funds rate eight times, from 1.00% to 3.00%. However, during this time longer-term Treasury yields fell, with the ten-year note’s yield declining from 4.69% to 4.00%. Chairman Alan Greenspan testified before the U.S. Congress that this pattern presented a “conundrum.” According to our model, this decline in the ten-year yield can be apportioned almost evenly between reductions in ten-year expected inflation (28 basis points) and the ten-year real yield (27 basis points), as well as a fall in the 10-year inflation risk premium (12 basis points). The economic intuition from the model is that the Federal Reserve’s tightening increased investor confidence in monetary discipline by lowering inflation’s “central tendency.” Moreover, a reduction monetary uncertainty proxied by a less volatile central tendency reduced both real and inflation risk premia.

Our completely affine model of nominal and real term structures differs from others by having four sources of uncertainty but seven state variables. The
model’s factors include the short-term real rate, the short-term rate of expected inflation, and inflation’s central tendency. These factors largely determine the cross-section of bond yields, whereas their innovations, along with innovation in actual inflation, drive four additional volatility-state variables. The volatility factors follow the nonlinear asymmetric GARCH model of Engle and Ng (1993) and exclusively determine bond risk premia that vary over time and can even change signs. Our study also breaks new ground by estimating the model using data on nominal yields, and survey forecasts of inflation, and also inflation swap rates.

Because of the Federal Reserve’s policy of pegging short-term nominal rates, the short-term real rate is typically highly negatively correlated with changes in short-run expected inflation. Short-term real rates are quite volatile and were negative from 2002 to 2005, which may have led to an overexpansion of credit. However, inflation’s longer-run central tendency trended downward over our 1982-to-2010 sample period and continues to be low, despite a re-emergence of negative real rates.

We also find that real risk premia at short maturities increase when the volatility of the short-term real interest rate is high, but greater longer-maturity real risk premia are generated mainly by greater volatility from inflation’s central tendency, consistent with real costs of longer-run inflation uncertainty. Inflation risk premia at short maturities are inversely related to the volatility of unanticipated inflation. However, longer-run inflation risk premia respond positively to greater volatility from inflation’s central tendency. However, during periods of financial crisis, we find that the “flight-to-quality” attraction for longer-run Treasuries can be linked to unexpected deflation that can substantially lower nominal bonds’ excess returns via a decline in their inflation risk premia.

Finally, comparing our model’s implied yields for inflation-indexed bonds to those of TIPS suggests that TIPS bore a large liquidity premium prior to 2004. Perhaps because of the 2003 introduction of inflation derivatives, such as inflation swaps, arbitrage possibilities may have eliminated TIPS mispricing until the middle of 2008. However, after the Lehman Brothers’ bankruptcy in September 2008, the link between nominal Treasuries, TIPS, and inflation swaps was broken, and a very large liquidity premium for TIPS reappeared.

Appendix

Proof of Proposition 1

The dynamics for $\pi_t$, $r_t$, and $\alpha_t$ are straightforward. The interesting diffusion limits are for the $h_{j,t}$ state variables. Substitute the expressions for $d_{j,i}$, $i = 0, \ldots, 3$, given in Proposition 1, into the last line of Equation (6). Upon simplification, this leads to

$$
\Delta h_{j,t} = (\kappa_j \theta_j + \frac{1}{4} v_{j}^2) \Delta t + \frac{1}{4} v_{j}^2 (\epsilon_{j,t+\Delta t}^2 - 1) \Delta t + h_{j,t}^2 \kappa_j \Delta t (\frac{\kappa_j \Delta t}{4} - 1)
- v_j (1 - \frac{\kappa_j v_j}{2} \Delta t) \sqrt{\Delta t} \epsilon_{j,t+\Delta t}.
$$

(A1)
Taking limit of this expression, the second term converges to zero, which leads to the result.

**Lemma 1.** Let $X \sim N(0, 1)$. Then for $Q_2 > -\frac{1}{2}$, we have

$$E\left[ e^{Q_1 X - Q_2 X^2} \right] = \exp\left[ \frac{Q_1^2}{2(1 + 2Q_2)} - \frac{1}{2} \ln(1 + 2Q_2) \right].$$

**Proof:**

The expectation can be written as $E\left[ e^{Q_1 X - Q_2 X^2} \right] = \int_{-\infty}^{\infty} e^{Q_1 x - Q_2 x^2 - \frac{1}{2}x^2} dx$, and the result follows after completing the square and using properties of the normal density.

**Proof of Proposition 2**

The proof of the first result follows by substituting the nominal pricing kernel into the bond pricing equation, $P_{n,t}^{(n)} = E_t \left[ \frac{M_{t+\Delta t}^{n+1}}{M_t^n} P_{n,t+\Delta t}^{(n-1)} \right]$, to obtain

$$P_{n,t}^{(n)} = E_t \left[ e^{-(\tau_t + r_t - \phi_1 h^2_{t,j}) \Delta t - \frac{1}{2} \sum_{j=1}^{4} \phi^2_{j,h_{j,t}} \Delta t - \sum_{j=1}^{4} \phi_{j,h_{j,t}} \sqrt{\Delta t} \epsilon_{j,t+\Delta t} P_{n,t+\Delta t}^{(n-1)}} \right]. \quad (A2)$$

Now assume the bond price has the form of Equation (18), and substitute it into the left- and right-hand sides of Equation (A2). Substituting in for the state variables at date $t + \Delta t$ using Equation (6), collecting all coefficients of the random variables of the same type together, and then taking expectations using Lemma 1 leads to the resulting recursive equations for the coefficients. The initial boundary conditions come from considering the case in which $n = 1$. The recursive equations are

$$K_{n+1} = K_n + (b_0 B_n + c_0 C_n + \sum_{j=1}^{4} d_{j,1} D_{j,1,n}) \Delta t$$

$$A_{n+1} = \Delta t + (1 + a_2 \Delta t) A_n + b_2 B_n \Delta t$$

$$B_{n+1} = \Delta t + a_1 \Delta t A_n + (1 + b_1 \Delta t) B_n$$

$$C_{n+1} = A_n \Delta t + (1 + c_1 \Delta t) C_n$$

$$D_{j,n+1} = \left( -\phi_1 I_{(j=1)} + \frac{\phi^2_{j,1}}{2} \right) \Delta t + D_{j,n} \left( 1 + (d_{j,1} + d_{j,2} d_{j,3}^2) \Delta t \right)$$

$$- \frac{Q^2_{j,n}}{2(1 + 2D_{j,n} d_{j,2} \Delta t)}, \quad (A3)$$

where $I_{(j=1)}$ is the indicator function equal to 1 if $j = 1$ and is 0 otherwise, and

$$Q_{1,1,n} = (\phi_1 + A_n \beta_1 + B_n \gamma_1 + C_n \rho_1 - D_{1,n} 2d_{12} d_{13} \Delta t) \sqrt{\Delta t}$$

$$Q_{2,2,n} = (\phi_2 + A_n \beta_2 + B_n \gamma_2 + C_n \rho_2 - D_{2,n} 2d_{21} d_{23} \Delta t) \sqrt{\Delta t}$$

$$Q_{3,3,n} = (\phi_3 + B_n \gamma_3 + C_n \rho_3 - D_{3,n} 2d_{32} d_{33} \Delta t) \sqrt{\Delta t}$$

$$Q_{4,4,n} = (\phi_4 + C_n \rho_4 - D_{4,n} 2d_{42} d_{43} \Delta t) \sqrt{\Delta t}. \quad (A4)$$

Next consider the pricing of TIPS. Let $t$ be the current date, $t_e = t + (n - d) \Delta t$ and $t_p = t_e + d \Delta t$. We will compute $V_{t,t_s}^{(n,d)} = P_{R,t}^{(n,d)} I_t / I_{t_s}$. Now, assume the structure in Equation (19) of Proposition 2:

$$V_{t,t_s}^{(n,d)} = \frac{I_t}{I_{t_s}} e^{-K_n - A_n \tau_t - B_n r_t - C_n \alpha_t - \sum_{j=1}^{4} \tilde{D}_{j,n} h^2_{j,t}}. \quad (A5)$$
Now,

\[ V_{t,s}^{(n,d)} = E_t \left[ \frac{M_{t+\Delta t}}{M_t} V_{t+\Delta t,s}^{(n-1,d)} \right]. \]  

(A6)

Substituting in the structure for \( V_{t,s}^{(n-1,d)} \) leads to

\[ V_{t,s}^{(n,d)} = E_t \left[ \frac{I_{t+\Delta t}}{I_t} \frac{M_{t+\Delta t}}{M_t} e^{-\tilde{K}_{n+1} - \tilde{A}_{n+1} - \tilde{B}_{n+1} - \tilde{C}_{n+1} + \sum_{j=1}^{4} \alpha_j \tilde{D}_{j,n+1} \Delta t} \right]. \]  

(A7)

This can be rewritten as

\[ V_{t,s}^{(n,d)} = \frac{I_t}{I_s} E_t \left[ \frac{I_{t+\Delta t}}{I_t} \frac{M_{t+\Delta t}}{M_t} e^{-\tilde{K}_{n+1} - \tilde{A}_{n+1} - \tilde{B}_{n+1} - \tilde{C}_{n+1} + \sum_{j=1}^{4} \alpha_j \tilde{D}_{j,n+1} \Delta t} \right]. \]  

(A8)

Substituting for the nominal pricing kernel and the inflation process using Equations (1) and (2), as well as for the state variables at date \( t + \Delta t \) using Equation (6), collecting all coefficients of the random variables of the same type together, taking expectations using Lemma 1, and using Equation (13) leads to the recursive equations for the coefficients.

The boundary conditions are obtained by recognizing that at date \( t + (n - d) \Delta t \), the final payment is known but is deferred by \( d \) periods. So, the boundary conditions with \( d \) periods to go are given by the known payment multiplied by the \( d \)-period nominal bond price. The recursive equations for real bonds are

\[
\begin{align*}
\tilde{K}_{n+1} &= \tilde{K}_n + (b_0 \tilde{B}_n + c_0 \tilde{C}_n) + \sum_{j=1}^{4} \alpha_j \tilde{D}_{j,n+1} \Delta t \\
&\quad + \frac{1}{2} \sum_{j=1}^{4} \ln(1 + 2d_j \tilde{D}_{j,n+1} \Delta t)
\end{align*}
\]

\[
\tilde{A}_{n+1} = (1 + a_2 \Delta t) \tilde{A}_n + b_2 \Delta t \tilde{B}_n
\]

\[
\tilde{B}_{n+1} = (1 + a_1 \tilde{A}_n) \Delta t + (1 + b_1 \Delta t) \tilde{B}_n
\]

\[
\tilde{C}_{n+1} = \Delta t \tilde{A}_n + (1 + c_1 \Delta t) \tilde{C}_n
\]

\[
\tilde{D}_{j,n+1} = \left( \frac{1}{2} - \phi_1 \right) I_{j=1} \Delta t + \frac{1}{2} \phi_j^2 \Delta t + \tilde{D}_{j,n} \left( 1 + \left( d_{j,1} + d_{j,2} d_{j,3}^2 \right) \Delta t \right)
\]

\[
\frac{\tilde{Q}_{2,n}^2}{2(1 + 2 \tilde{D}_{j,n+1} d_{j,2} \Delta t)}.
\]

where \( I_{j=1} \) equals 1 only if \( j = 1 \) and

\[
\begin{align*}
\tilde{Q}_{1,n} &= (1 - \phi_1 - \tilde{A}_n \beta_1 - \tilde{B}_n \gamma_1 - \tilde{C}_n \rho_1 + \tilde{D}_{1,n} 2d_{12}d_{13} \sqrt{\Delta t} \sqrt{\Delta t})
\end{align*}
\]

\[
\begin{align*}
\tilde{Q}_{2,n} &= (-\phi_2 - \tilde{A}_n \beta_2 - \tilde{B}_n \gamma_2 - \tilde{C}_n \rho_2 + \tilde{D}_{2,n} 2d_{22}d_{23} \sqrt{\Delta t} \sqrt{\Delta t})
\end{align*}
\]

\[
\begin{align*}
\tilde{Q}_{3,n} &= (-\phi_3 - \tilde{B}_n \gamma_3 - \tilde{C}_n \rho_3 + \tilde{D}_{3,n} 2d_{32}d_{33} \sqrt{\Delta t} \sqrt{\Delta t})
\end{align*}
\]

\[
\begin{align*}
\tilde{Q}_{4,n} &= (-\phi_4 - \tilde{C}_n \rho_4 + \tilde{D}_{4,n} 2d_{42}d_{43} \sqrt{\Delta t} \sqrt{\Delta t})
\end{align*}
\]

(A10)

**Proof of Proposition 3**

Note that \( E_t \ln(I_{t+n\Delta t}/I_t) = E_t \ln(I_{t+n\Delta t}/I_t) + \ln(I_{t+n\Delta t}/I_{t+\Delta t}) \). Assuming the exponential affine structure in Proposition 3 with \((n - 1)\) periods to go, then
\[ E_t \left[ \ln \left( \frac{I_{t+n\Delta t}}{I_t} \right) \right] = E_t \left[ \ln \left( \frac{I_{t+\Delta t}}{I_t} \right) + K_{n-1}^* + A_{n-1}^* \Delta t + B_{n-1}^* \Delta t \right. \\
\left. + C_{n-1}^* a_{t+\Delta t} + \sum_{j=1}^4 D_{j,n}^* h_{j,t+\Delta t} \right], \]  

(A11)

Substitute in the dynamics for inflation from Equation (2) and compute the resulting expectation. The recursive equations in turn are

\[ K_{n+1}^* = K_n^* + (b_0 B_n^* + c_0 C_n^* + \sum_{j=1}^4 (d_{j0} + d_{j2}) D_{j,n}^*) \Delta t \]
\[ A_{n+1}^* = \Delta t + (1 + a_2 \Delta t) A_n^* + b_2 \Delta t B_n^* \]
\[ B_{n+1}^* = a_1 \Delta t A_n^* + (1 + b_1 \Delta t) B_n^* \]
\[ C_{n+1}^* = \Delta t A_n^* + (1 + c_1 \Delta t) C_n^* \]
\[ D_{j,n+1}^* = D_{j,n}^* \left[ 1 + (d_{j1} + d_{j3} d_{j3}^-) \Delta t \right] - 1_{(j=1)} \frac{1}{2} \Delta t. \]

(A12)

**Formulae for Expected Excess Returns and Yield Covariances**

For zero-coupon bonds of maturity \( n \) periods, the expected excess nominal return on a nominal bond and the expected excess real return on a real bond are

\[ \tau_{n,j}^{(n)} = \frac{1}{\Delta t} \sum_{j=1}^4 \left( v_{j0}^{(n)} + v_{j1}^{(n)} h_{j,t}^2 \right), \]
\[ \tau_{R,j}^{(n)} = \frac{1}{\Delta t} \sum_{j=1}^4 \left( v_{j0}^{*(n)} + v_{j1}^{*(n)} h_{j,t}^2 \right), \]

(A13)

(A14)

where

\[ v_{j0}^{(n)} = \frac{1}{2} \ln(1 + 2D_{j,n-1} d_{j2} \Delta t) - d_{j2} D_{j,n-1} \Delta t \]
\[ v_{j1}^{(n)} = \frac{1}{2} \Delta t - \phi_j^2 \Delta t - \frac{Q_{j,n-1}^2}{2(1 + 2D_{j,n-1} d_{j2} \Delta t)} \]
\[ v_{j0}^{*(n)} = \frac{1}{2} \ln(1 + 2\tilde{D}_{j,n-1} d_{j2} \Delta t) - d_{j2} \tilde{D}_{j,n-1} \Delta t \]
\[ v_{j1}^{*(n)} = \left( \frac{1}{2} - \phi_j^1 \right) I_1 \Delta t + \frac{1}{2} \Delta t - \frac{Q_{j,n-1}^2}{2(1 + 2\tilde{D}_{j,n-1} d_{j2} \Delta t)}. \]

where \( I_1 \) equals 1 if \( j = 1 \) and is 0 otherwise. The covariance of the changes in yields for \( m \)-period and \( k \)-period maturity nominal and real bonds are

\[ Cov_t \left( \gamma_{N,t+\Delta t}^{(m)} - \gamma_{N,t}^{(m)}, \gamma_{N,t+\Delta t}^{(k)} - \gamma_{N,t}^{(k)} \right) = \frac{\sum_{j=1}^4 \left( a_{j,m} a_{j,k} h_{j,t}^2 \Delta t + 2D_{j,m} D_{j,k} d_{j2} \Delta t^2 \right)}{m k \Delta t^2}. \]

(A15)

\[ Cov_t \left( \gamma_{R,t+\Delta t}^{(m)} - \gamma_{R,t}^{(m)}, \gamma_{R,t+\Delta t}^{(k)} - \gamma_{R,t}^{(k)} \right) = \frac{\sum_{j=1}^4 \left( a_{j,m} a_{j,k} h_{j,t}^2 \Delta t + 2D_{j,m} D_{j,k} d_{j2} \Delta t^2 \right)}{m k \Delta t^2}. \]

(A16)

where

\[ a_{1n} = \beta_1 A_n + \gamma_1 B_n + \rho_1 C_n - 2d_{12} d_{13} \sqrt{\Delta t} D_{1n} \]
\[ a_{2n} = \beta_2 A_n + \gamma_2 B_n + \rho_2 C_n - 2d_{22} d_{23} \sqrt{\Delta t} D_{2n} \]
\[ a_{3n} = \gamma_3 B_n + \rho_3 C_n - 2d_{32} d_{33} \sqrt{\Delta t} D_{3n} \]
\[ a_{4n} = \rho_4 C_n - 2d_{42} d_{43} \sqrt{\Delta t} D_{4n}, \]
and the $a_{jn}^*$ values have the same form as $a_{jn}$ values, except that the $A_n$, $B_n$, $C_n$, and $D_{jn}$ values are replaced by $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_{jn}$ values, respectively.

References


