The Dynamic Pricing Problem
from a Newsvendor's Perspective

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Abstract

The dynamic pricing problem concerns the determination of selling prices over time for a product whose demand is random and whose supply is fixed. We approach this problem in a novel way by formulating a dynamic optimization model in which the demand function is iso-elastic but the random demand process is quite general. Ultimately, what we find is a strong parallel between the dynamic pricing problem and dynamic inventory models. This parallel leads to a reinterpretation of the dynamic pricing problem as a price-setting newsvendor problem with recourse, which is useful not only because it yields insights into the optimal solution, but also because it leads to additional insights into how pricing recourse affects the actions and profits of a price-setting newsvendor. We make contributions in three areas: First, we develop structural properties that define an optimal pricing strategy over a finite horizon and investigate how that policy impacts a newsvendor’s optimal procurement policy and optimal expected profit. Second, we establish a practical and efficient algorithm for computing the optimal prices. Third, we examine how market parameters affect the optimal solution through a series of numerical experiments that utilize the algorithm.

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1 Introduction

The dynamic pricing problem concerns the determination of selling prices over time for a product whose demand is random and whose supply is fixed. This problem tends to present significant computational challenges. To overcome these difficulties, heuristics are often employed to compute solutions that may not be optimal. In general, the problem is to dynamically adjust selling prices, as the fixed inventory is depleted, to maximize the expected revenue stream over a finite planning horizon. A prototypical example is the pricing of seats on an airline flight: although an aircraft is committed to a flight in advance, the price of the seats can change dynamically right up until take-off. Other examples now include the pricing of hotel rooms, fashion goods, discontinued or left over products, golf course tee times, even financial aid packages (Virshup, 1997). The payoff functions in these problems are typically not concave, which creates the computational challenge.

Because of the strategic importance of a firm’s pricing decisions and the technical challenges inherent in the modeling and computation of optimal decisions, the dynamic pricing problem has generated a fair amount of interest in recent years among researchers across such fields as operations research, marketing, and economics. In this paper, we approach the problem in a novel way by formulating a dynamic program in which the relationship between demand and price is somewhat specific, but the random demand process is quite general. Our focus is three-fold: First, we develop structural properties that define an optimal pricing strategy over a finite horizon and investigate how that policy impacts the firm’s optimal procurement decision and optimal expected profit. Second, we establish a practical and efficient algorithm for computing the optimal prices for each period of the finite horizon. And third, by implementing the algorithms for a variety of problem instances, we examine how market parameters affect the optimal solution. Ultimately what we find is a strong parallel between the dynamic pricing problem and dynamic inventory models; thus, we can draw on the rich insights from inventory theory to better understand the structure of the dynamic pricing policy and its efficacy for higher-level managerial decision making.

We approach the problem within the following modeling framework: A retailer has a single opportunity to establish an inventory (or capacity) level prior to the start of a selling season that consists of multiple periods. Demand in each period is random and depends on price, but the elasticity of demand is known and is independent of price. At the beginning of each period, a price is announced that depends both on the amount of inventory that remains and on the number of periods that remain to sell it. Then demand
is realized, the period ends, and the next one begins.

Our model stipulates a specific class of demand functions, those functions in which price elasticity of demand is constant, but places no requirements on the form of the probability distribution that is used to characterize the uncertainty in demand. This model offers three benefits that, collectively, constitute its primary contribution to the literature. One benefit of our model is its tractability. In particular, we find that, with an appropriate transformation of variables, the dynamic pricing problem that we formulate can be reduced to a sequence of state-independent, static, single-variable optimization problems. Moreover, each of the resulting optimization problems is of the same form, differing from the others only by the magnitude of a single coefficient. As a result, determination of the optimal pricing policy does not require the solution of a dynamic program; and, the complexity involved in computing the optimal price for each period does not increase significantly as the number of periods comprising the finite horizon increase. The model's tractability also leads to new insights both on the parallels between the dynamic pricing problem and dynamic inventory problems, and on the value and effect that recourse flexibility (in the form of dynamic pricing) provides to a price-setting newsvendor.

A second benefit of our model is its robustness: it can be applied using any demand distribution. In contrast, most pricing models appearing in the yield management literature require demand to be characterized by the Poisson distribution in order to obtain tractability. One limitation of the Poisson approach is that it is a single-parameter distribution; hence there is no freedom to estimate separately the mean and variance (and higher moments) of demand. Another limitation is that the coefficient of variation of demand decreases as the mean of demand increases; hence, for applications in which mean demand is very large, demand uncertainty is “modeled away” and, as a result, the business environment is effectively treated as a deterministic one. A compound-Poisson distribution can be used to overcome these limitations, but then the resulting solution procedure typically becomes enough of an onus that heuristics need to be employed. Our model allows us to address these issues without having to revert to heuristics.

A third benefit of our model is that it has practical and intuitive appeal. Specifically, because it is formulated as a periodic pricing model, it can be applied directly to business scenarios in which it either is not practical or is not desirable to adjust the selling price after each sale of a unit of inventory (which might be the case, for example, for fashion goods). Yet, because the computational burden of the solution procedure does not increase significantly as the number of periods in the planning horizon is increased, our
model also can be applied to business scenarios in which it is acceptable practice to adjust the price after each unit sale (e.g., for airline seats) simply by defining the “periods” as appropriately small intervals of time. The intuitive appeal derives from the fact that properties of the optimal pricing policy are sensible. For example, our model yields the result that, everything else being equal, the optimal price to set in a given period is a decreasing function of the amount of stock that is available for sale in that period.

The implicit “cost” of our modeling approach is the assumption that demand elasticity is independent of price. However, this specification of price dependency is as common to the economics literature as the specification of a Poisson process is to the OR literature. Basically, by applying this modeling convenience to a dynamic pricing setting, we in effect are exchanging one dimension of tractability (Poisson demand uncertainty) with another dimension of tractability (iso-elastic demand). In return, we secure the benefits described above; ultimately, we end up with a state-space reduction technique that leads to a practical algorithm for actually computing the optimal price path. Thus, our model offers a useful alternative for actually computing the optimal price for each of an arbitrary number of periods and for exploring new insights, while still maintaining qualitative results that are consistent with the existing literature.

The remainder of this paper is organized as follows. In Section 2, we position the paper in relation to the literature. In Section 3, we formulate the dynamic decision problem and demonstrate the state-space reduction that allows the model to be solved as a series of static problems rather than as a dynamic program. Then, in Section 4, we establish the structure of the optimal pricing policy, discuss corresponding implications on the initial inventory decision, and develop corresponding insights. In Section 5, we develop properties to improve the efficiency of the solution procedure for the special case in which the uncertainty in demand has finite support. Then, in Section 6, we implement the algorithm to further investigate implications and interpretations of the optimal solution. We conclude the paper with Section 7.

2 Relationship to the Literature

The model developed in this paper spans several streams of literature. One stream is the literature on dynamic pricing. Single-product dynamic pricing models were first studied by Kincaid and Darling (1963), who formulated a continuous-time stochastic dynamic program and developed properties of
the revenue function. Gallego and van Ryzin (1994) later derived useful structural properties of the optimal price and proposed a deterministic heuristic for solving the problem; Zhao and Zheng (2000) then derived structural properties under more general conditions and found a closed form solution for the case of a discrete price set. Gallego and van Ryzin (1997) extended their single-product model and asymptotically-optimal heuristic to the multiple-product case. Common to these models is the formulation of the problem as a continuous-time stochastic program in which demand uncertainty is characterized by a Poisson process with a price-dependent intensity. The solutions for these models are both time and state dependent. In contrast, we approach the problem using a discrete-time model in which demand uncertainty can be characterized by a generic distribution; and we establish a solution procedure that is state independent. Bitran and Mondschein (1997) considered a discrete-time pricing model for a retail setting. However, they assumed that demand is Poisson and developed an optimal solution that is state dependent.

A second stream related to our model is the literature on dynamic inventory models with pricing and stochastic demand. Petruzzi and Dada (1999) provide a recent review of this literature, but representative papers most related to our model include Zabel (1972), Federgruen and Heching (1997), and Petruzzi and Dada (2002). Petruzzi and Dada (2002) were particularly instrumental in the development of our model because they specifically considered a case in which demand is modeled as a constant-elastic function of price. Their focus, however, is on learning the demand distribution when lost sales are not observable. And, like the other papers in this literature stream, their model differs from ours because it applies to scenarios in which inventory can be replenished each period.

More similar to our model is the model by Chan et al. (2001), who consider a manufacturing setting in which production occurs every period, but each period’s production decision is determined ahead of time, at the beginning of the finite horizon. Then, price is determined dynamically, at the start of each period. However, Chan et al. (2001) maintain discretion over each period’s inventory decision by allowing some or all of the predetermined production for a given period to be “set aside” for future periods. In effect, this discretion replaces production as the mechanism to achieve a desired inventory level in a given period. In addition, solving for the optimal prices in their setting quickly becomes computationally intractable; thus, like in many of the models in the first literature stream, a heuristic is proposed.

A third related literature stream is that on yield management, which is a set of problems that can be more generally classified as perishable asset
revenue management (PARM) models. The basic problem of yield management is how to sell a finite inventory over a finite horizon to maximize the total revenue. Models of this sort have developed a rich history in recent years. They typically are formulated by assuming that demand is segmented into classes, and that each class has associated with it a fixed price that is determined exogenously. However, there is a number of variations on this theme that appear in the literature. One variation involves settings in which demands for the different classes arrive sequentially. In these cases, the fundamental question that must be answered each time a demand occurs is whether to accept the demand or to reserve the unit of inventory for possible sale later to a potentially higher-paying customer. Models of this variety include Littlewood (1972), Belobaba (1989), Brumelle and McGill (1993), and Robinson (1995).

A second variation involves settings in which demands for different classes occur concurrently, and inventory can be made available to multiple classes simultaneously. In these cases, the key question is how much inventory to allocate to each demand class at any given time. Models of this type include Gerchak et al. (1985), Lee and Hersh (1993), Subramanian et al. (1999), and Zhao and Zheng (2001).

Finally, a third variation on this theme involves settings in which demands for different classes, in effect, occur concurrently, but inventory cannot be made available to multiple classes (at multiple prices) simultaneously. The basic decision in these models is when to switch from one demand class (usually characterized as higher paying customers) to another class (usually characterized as lower paying customers). Models of this type include Feng and Gallego (1995), Feng and Xiao (2000), and Petruzzi and Monahan (2002).

It is worth noting that this third variation also can be interpreted as a class of the dynamic pricing problem in which a discrete price set is given. Thus, this class of models bridges the dynamic pricing and yield management literatures. In general, however, yield management models differ from the dynamic pricing model because they assume that price is exogenous; thus, their primary focus, in effect, is on inventory control. McGill and van Ryzin (1999) offer an excellent survey of not only yield management and dynamic pricing models, but also of related models on airline overbooking and demand forecasting.
3 Model And Solution Procedure

A finite-horizon selling season consists of \( T \) periods, indexed so that period \( t \) represents the number of periods remaining in the selling season. Demand in period \( t \) is random and depends on price as follows: \( D_t(p) = A_t p^{-b} \), where \( b > 1 \) and \( A_1, \ldots, A_T \) are independent, identically distributed (iid) random variables, each with known cumulative distribution function (cdf) \( F \) and corresponding probability density function (pdf) \( f \). Notice that there are two key modeling elements associated with the demand process: the uncertainty effect and the price effect. We incorporate the price effect by assuming that the elasticity of demand is equal to the constant \( b \) and is therefore independent of price. (See, for example, Petruzzi and Dada (1999).) In addition, we incorporate the uncertainty effect by assuming that the randomness in demand is price-independent and multiplicative in nature. (See for example, Karlin and Carr (1962).) Note that the stationary distribution \( F \) can be replaced by the non-stationary process \( F_t \) without affecting the analysis or results in a material way. We discuss the implications of this replacement in Section 7.

Prior to the beginning of the season, an initial stock of \( S \) units is acquired (e.g., procured, produced) at a per-unit cost of \( c \) and is made available for sale over the finite horizon. At the beginning of each period, as long as some of the initial stock remains, a selling price is chosen. If no stock remains as of the beginning of period \( t \), the dynamic pricing problem ends. In general, \( p_t \), the selling price set for period \( t \), depends both on how much of the initial stock remains as of the beginning of period \( t \) and on how many selling periods still remain.

Let \( R_t(I_t) \) denote the maximum expected revenue-to-go function as of the beginning of period \( t \), given that \( I_t \) is the stock remaining as of the beginning of period \( t \), and that an optimal dynamic pricing policy is followed for the remaining \( t \) periods. The observation that \( I_t \) is equivalent to the number of leftovers that remain at the end of period \( t + 1 \) leads to a recursive formula for \( R_t(I_t) \):

\[
R_t(I_t) = \max_p \left\{ p \left[ \text{period-} t \text{ sales} \right] + \mathbb{E} \left[ R_{t-1}(\text{period-} t \text{ leftovers}) \right] \right\}
= \max_p \left\{ p \left( I_t - \mathbb{E} \left[ A_t p^{-b} \right] \right) + \mathbb{E} \left[ R_{t-1} \left( \left[ I_t - A_t p^{-b} \right]^+ \right) \right] \right\},
\]

where \( I_T = S \) and \( R_0(I) = 0 \) for all \( I \).

Let \( p^*_t \) denote the optimal price for period \( t \) and \( S^* \) denote the optimal
starting stock level. Then (1) implies that
\[ p^*_t = \arg\max_p \left\{ p \left( I_t - E \left[ I_t - A_t p^{-b} \right]^+ \right) + E \left[ R_{t-1} \left( \left[ I_t - A_t p^{-b} \right]^+ \right) \right] \right\} \] (2)
\[ S^* = \arg\max_S \{ R_T(S) - cS \} \] (3)

Notice from (2) that the optimization problem required to compute \( p^*_t \) depends on \( I_t \), the state of the system as of the beginning of period \( t \). This problem, however, can be simplified to a state-independent optimization problem through the following transformation of variables. Let \( z_t = I_t / p^{-b}_t \) denote the period-\( t \) stocking factor (Petruzzi and Dada (1999)). Then, the period-\( t \) expected sales function can be written as the product of \( I_t \) and a sales factor that is independent of \( I_t \):
\[ E \left[ \text{period-}t \text{ sales} \right] = I_t - E \left[ I_t - A_t p^{-b} \right]^+ = I_t \left( z_t - \frac{[z_t - A_t]^+}{z_t} \right) \] (4)

As a result, \( R_t(I_t) \), the maximum expected-revenue-to-go-function, can be written as a multiplicatively separable function of \( I_t \), which we demonstrate by the following proposition.

**Proposition 1.** Let \( m = 1 - 1/b \) serve as a proxy for the elasticity of demand. Moreover, let \( r^*_t = \max_z r_t(z) \) be defined as a constant that can be interpreted as an optimal revenue factor, where \( r^*_0 = 0 \) and
\[ r_t(z) = \frac{z - E \left[ (z - A_t)^+ \right] + r^*_{t-1} E \left[ ((z - A_t)^+)^m \right]}{z^m} \] (5)

Then, \( R_t(I_t) = r^*_t I_t^m \).

**Proof.** The proof follows by induction on \( t \), given the induction hypothesis \( R_t(I_t) = r^*_t I_t^m \). If \( t = 1 \), then, from (1), (4), the definition of \( z_t \), and (5),
\[ R_1(I_1) = \max_z \left\{ \left( \frac{I_1}{z} \right)^m \left( z - E \left[ (z - A_1)^+ \right] \right) \right\} = I_1^m \max_z r_1(z) = r^*_1 I_1^m, \]
which establishes that the result is true for \( t = 1 \). Assume that the induction hypothesis is true for \( t = i \), so that \( R_i(I_i) = r^*_i I_i^m \), and consider the case \( t = i + 1 \). From (1) and the induction hypothesis,
\[ R_{i+1}(I_{i+1}) = \max_z \left\{ \left( \frac{I_{i+1}}{z} \right)^m \left( z - E \left[ (z - A_{i+1})^+ \right] \right) + E \left[ R_i \left( I_{i+1} \left( \frac{z - A_{i+1}}{z} \right) \right) \right] \right\} \]
\[ = I_{i+1}^m \cdot \max_z \left\{ \frac{z - E(z - A_{i+1})^+ + r^*_i E \left[ ((z - A_{i+1})^+)^m \right]}{z^m} \right\} = r^*_i I_{i+1}^m. \]
Thus, if the induction hypothesis is true for \( t = i \), then it is true for \( t = i + 1 \). Since it is true for \( t = 1 \), it is therefore true for all \( t \), which completes the proof.

Thus, the computation of \( R_t(I_t) \) requires only a maximization of \( r_t(z) \), a function that does not depend upon \( I_t \). Let \( z_t^* = \arg \max_z r_t(z) \) denote the optimal stocking factor for period \( t \). Clearly this value depends on \( t \), the number of periods remaining in the season, but is independent of \( I_t \), the number of items available for sale in period \( t \). Consequently, \( z_t^* \) can be computed for all \( t \) at the beginning of the finite horizon, without first having to observe \( I_t \), by iteratively solving \( T \) “single-period” problems using the following algorithm: First set \( r_0 = 0 \). Then, for \( t = 1, \ldots, T \), find \( z_t^* = \arg \max_z r_t(z) \) and set \( r_t^* = r_t(z_t^*) \), where \( r_t(z) \) is given by (5). In the worst case, maximizing the function \( r_t(z) \) requires an exhaustive search over \( z \)’s domain. However, in Sections 4 and 5 we develop properties of \( r_t(z) \) that give rise to more efficient search algorithms for computing \( z_t^* \).

Given the state-independent sequence of optimal stocking factors \( z_t^* \), the optimal price for period \( t \) can be recovered from the definition of \( z \), once \( I_t \) is observed:

\[
p_t^* = \left( \frac{z_t^*}{I_t} \right)^{1-m}.
\]  

(6)

Thus, the optimal price in period \( t \) is decreasing and convex as a function of \( I_t \), the amount of inventory that remains at the time that the period-\( t \) price is chosen (because \( 0 < m < 1 \)). This intuitive result helps validate the model. Moreover, from Proposition 1, \( R_T(S) = r_T^* S^m \), which is increasing and concave in \( S \). Therefore, (3) implies

\[
S^* = \left( \frac{m r_T^*}{c} \right)^{1/(1-m)} = \left( \frac{m r_T^*}{c} \right)^{b}
\]  

(7)

and

\[
r_T^* \cdot (S^*)^m - cS^* = \frac{1-m}{m} c S^* = \frac{1-m}{m} c \left( \frac{m r_T^*}{c} \right)^{b},
\]  

(8)

which provide convenient closed-form expressions for the retailer’s optimal stocking level and optimal expected profit for the season. Notice that \( S^* \) is decreasing and convex as a function of \( c \), which again provides intuitive validation for the model.
4 Properties Of The Optimal Solution

In Section 3, we demonstrated how the dynamic pricing problem can be reduced to an iterative procedure involving the solution of $T$ single-variable optimization problems by reformulating the problem as a dynamic safety factor problem. The result was an independence between periods that is such that the $t$th iteration of the problem can be constructed from the $t-1$st iteration of the problem by replacing a single constant term from the $t-1$st iteration with the computed solution of the $t-1$st iteration.

In this section, we establish and discuss the following properties of the retailer’s optimal dynamic pricing problem, which are useful for developing insights as well as more efficient solution procedures:

1. If the distribution of $A_t$ has an increasing generalized hazard rate (IGFR), then the optimal stocking factor for the final period of the selling season is characterized uniquely by an implicit function.

2. The optimal stocking factor is increasing in the number of periods remaining in the selling season.

3. If the random variable $A_t$ is rescaled to $n A_t$, the resulting optimal stocking factor for period $t$ will become rescaled by the same factor $n$.

4. The optimal stocking level is at least as large as the optimal stocking level for the price-setting newsvendor problem, which serves as a benchmark case that is otherwise equivalent to the dynamic pricing problem except that the pricing decision is not a dynamic one, but instead is made once, at the beginning of the selling season.

5. For the special case of deterministic demand, the optimal pricing policy is a single-price policy.

**Proposition 2.** Let $g(a) = a f(a) / [1 - F(a)]$ denote the generalized failure rate. If $dg(a)/da > 0$, then $z^*_1$, the optimal stocking factor for the last period of the selling season, is the unique solution to the following equation:

$$
\frac{z [1 - F(z)]}{z - \Lambda(z)} = m,
$$

where $\Lambda(z) = \int_0^z F(a) \, da$.  


Proof. By definition, \( z_1^* = \arg \max_z r_1(z) \), where, from (5), \( r_1(z) = (z - \Lambda(z))/z^m \). Thus, the first-order condition is

\[
\frac{dr_1(z)}{dz} = \frac{z [1 - F(z)] - m [z - \Lambda(z)]}{zm+1} = 0,
\]

which is satisfied if and only if \( z [1 - F(z)] / [z - \Lambda(z)] = m \). Thus, if we let

\( L(z) = z [1 - F(z)] / [z - \Lambda(z)] \),

then the proof is complete if we show that \( L(z) = m \) has exactly one solution. To that end, consider the behavior of \( L(z) \):

i. \( \lim_{z \to 0} L(z) = 1 > m \);

ii. \( \lim_{z \to \infty} L(z) = 0 < m \);

iii. \[
\frac{dL(z)}{dz} = \frac{[z - \Lambda(z)] [1 - F(z) - zf(z)] - z [1 - F(z)]^2}{[z - \Lambda(z)]^2} = \frac{L(z)}{z} [1 - g(z) - L(z)];
\]

iv. \[
\frac{d^2L(z)}{dz^2} \bigg|_{dL(z)/dz = 0} = \frac{-L(z)}{z} g'(z) < 0.
\]

From (iv), \( dL(z)/dz \) can change sign at most once, from positive to negative. But, from (iii), \( \lim_{z \to 0} (dL(z)/dz) \leq 0 \). Thus, (iii) and (iv) together imply that \( L(z) \) is a decreasing function of \( z \). Moreover, from (i) and (ii), \( L(z) > m \) for some range of \( z \), and \( L(z) < m \) for some range of \( z \). Therefore, (i) - (iv) together imply that \( L(z) = m \) has exactly one solution, namely \( z_1^* \).

Lariviere (1999) establishes the robustness of the IGFR condition: IGFR applies to many common classes of probability distributions including, but not limited to, the gamma, Weibull, and normal distributions. Thus, Proposition 2 is important because it ensures an efficient start to the iterative procedure used to compute the sequence of optimal stocking factors under quite general conditions. Proposition 3, which is presented next, provides a useful complement to Proposition 2 by establishing a monotone relationship between the optimal stocking factors.

**Proposition 3.** \( z_t^* > z_{t-1}^* \) for all \( t \).

**Proof.** The proof is by induction on \( t \), given the following two induction hypotheses:
i. \( r_{t+1}^* > r_t^* \); and

ii. \( z_{t+1}^* > z_t^* \).

To begin, notice from (5) that

\[
rt(z) = \int_0^z \left(1 - \frac{a}{z}ight)^m f(a) \, da;
\]

thus,

\[
dr_t(z) = dr(z) + m(r_t^* - r_{t-1}^*) \int_0^z \left(\frac{a}{z^2} \left(\frac{z}{z-a}\right)^{1-m} f(a) \, da.
\]

If \( t = 1 \), then, by definition, \( r_0(z) = 0 \) for all \( z \). Thus, from (9),

\[
r_1^* \geq r_2(z_1^*) = r_1(z_1^*) + r_1^* \int_0^{z_1^*} \left(1 - \frac{a}{z_1^*}\right)^m f(a) \, da > r_1(z_1^*) = r_1^*,
\]

where \( r_2^* \geq r_2(z_1^*) \) because \( r_2^* \geq r_2(z) \) for all \( z \), by the definition of \( r_2^* \). This establishes that induction hypothesis (i) is true for \( t = 1 \). Moreover, from (10), \( dr_2(z)/dz > dr_1(z)/dz \) for all \( z \); thus, induction hypothesis (ii) is also true for \( t = 1 \). Therefore, assume that both induction hypotheses are true for \( t = i \) and consider the case \( t = i + 1 \).

From (9), the definition of optimality, and induction hypothesis (i),

\[
r_{i+1}^* \geq rt(z_{i+1}^*) > rt(z_i^*) = r_i^*.
\]

And, from (10) and induction hypothesis (i), \( dr_{i+1}(z)/dz > dr_i(z)/dz \), which implies that \( z_{i+1}^* > z_i^* \). Thus, induction hypotheses (i) and (ii) are true for \( t = i + 1 \) if they are true for \( t = i \). Since they are true for \( t = 1 \), they are true for all \( t \), which completes the proof.

By establishing a monotone relationship between the optimal stocking factors, Proposition 3 provides a useful lower bound for the iteration procedure that serves as the general search algorithm for the optimal pricing policy. Accordingly, Proposition 3 can be applied together with Proposition 2 as follows: Beginning with the value of \( z_1^* \), which, from Proposition 2, can be computed easily under rather general conditions, the search procedure for determining \( z_2^* \) need only consider values of \( z \) that are such that \( z > z_1^* \). Then, for \( t = 2, \ldots, T \), the procedure is repeated iteratively, each time using \( z_{t-1}^* \) as the lower bound for the search for \( z_t^* \). Moreover, with each iteration, the search requires less effort because it is conducted over a progressively smaller region of \( z \) values.

From a managerial perspective, Proposition 3 and (6) establish that, for a fixed inventory level, the optimal price is increasing as a function of the
amount of time remaining in the selling season, a characteristic that is intu-
itive and is consistent with results of related dynamic pricing models (e.g.,
Gallego and van Ryzin (1997)). More generally, however, Proposition 3 estab-
lishes that the optimal stocking factor decreases as the number of periods re-
mainin in the season decreases. In our setting, the stocking factor is a proxy 
for safety stock. Thus, Proposition 3 can be interpreted to mean that it is op-
timal to carry less safety stock as the end of the selling season approaches. 
This is completely consistent with stochastic inventory theory, which indi-
cates that if price is fixed at the beginning of a finite selling season (usually 
exogenously), but the stocking quantity can be adjusted at the beginning 
of each new period, the optimal stocking quantity (which, because price is 
s fixed, serves as a proxy for the optimal safety stock) decreases as the number 
of periods remaining decreases. Intuitively, if a leftover occurs when there 
are many periods remaining in the finite horizon, there is a greater chance 
that the leftovers will eventually be sold. However, the chance of eventually 
selling a given leftover diminishes over time. Consequently, as the number 
of remaining periods decreases, the average cost associated with having a 
leftover increases, thereby resulting in a lower optimal safety stock. This in 
turn translates into a lower optimal stocking quantity in a dynamic inven-
tory problem and into a lower optimal stocking factor in a dynamic pricing 
problem. This parallel with inventory theory is another appealing feature 
that helps validate the model.

It is interesting to note that if the $A_t$ were not identically distributed, then 
Proposition 3 is not necessarily true. We demonstrate this by example with 
the following illustration.

**Illustration** Suppose $T = 2$ and $m = 0.5$. Let $A_1 \sim \text{uniform}(0,100)$ and 
$A_2 \sim \text{uniform}(0,10)$. Then,

$$z [1 - F(z)] = \frac{z(1 - z/100)}{z - z^2/200} = \frac{200 - 2z}{200 - z}.$$ 

Thus, from Proposition 2, $z_1^* = 200(1 - m)/(2 - m) = 66.667$. Correspond-
ingly, from (5), $r_1^* = 5.443$; and

$$r_2(z) = \begin{cases} 
\sqrt{z} \left(1 - \frac{z}{20}\right) + \frac{zr_1^*}{15} & \text{if } z < 10 \\
\frac{5}{\sqrt{z}} + \frac{zr_1^*}{15} \left(1 - \left(\frac{z - 10}{z}\right)^{1.5}\right) & \text{if } z \geq 10,
\end{cases}$$

which is maximized at $z_2^* = 36.432$. Thus, for this example, $z_2^* < z_1^*$. 

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4 PROPERTIES OF THE OPTIMAL SOLUTION
This illustration demonstrates that if the increase in demand (over time) is large enough, a future optimal stocking factor (e.g., \(z_t^*\)) can be greater than a nearer-term stocking factor (e.g., \(z_{t-1}^*\)). Thus, if \(A_{t-1}\) is large relative to \(A_t\), it may be that the optimal price in period \(t\) is higher than the optimal price in period \(t-1\), even if the available inventory levels are the same for both periods. Although this is not necessarily intuitive, it is again consistent with inventory theory: given a fixed price, if the distribution of demand is not stationary, then the optimal stocking quantities need not necessarily decrease as the number of periods remaining in the finite horizon decreases. Basically, if the standard deviation of demand for a future period is greater than the standard deviation of demand for a nearer-term period (as is the case in the above illustration), then the resulting upward pressure on safety stock could outweigh the downward pressure created by the reduced overage cost (associated with the future period).

From a technical standpoint, however, the possibility of an ambiguous relationship between subsequent periods’ optimal stocking factors is, at first glance, troubling: it means that Proposition 3 cannot be exploited to simplify the solution procedure for the iterative computation of the optimal stocking factors when the \(A_t\) are not identically distributed. Fortunately, however, the general search procedure still applies if the \(A_t\) are not stationary. (It just requires more effort to implement.) Moreover, for non-stationary \(A_t\), a rescaling of time such that the “periods” of the fixed-length selling season are defined as equal-spaced demand epochs rather than as equal-spaced time epochs might be possible. We briefly discuss this idea in Section 7.

Proposition 4 provides additional insight.

**Proposition 4.** Suppose \(D_t(p) = \tilde{A}_t p^{-b}\), where \(\tilde{A}_t = nA_t\) and define \(\tilde{z}_t^*\) as the corresponding optimal stocking factor for period \(t\). Then \(\tilde{z}_t^* = nz_t^*\) for all \(t\).

**Proof.** First, note that, by definition, \(\tilde{z}_t^* = \arg\max_z \hat{r}_t(z)\), where, analogous to (5),

\[
\hat{r}_t(z) = \frac{z - E[(z - \tilde{A}_t)^+] + \tilde{r}_{t-1} E\left[\left((z - \tilde{A}_t)^+\right)^m\right]}{z^m}
\]

\[
= \frac{1}{z^m} \left[ z - \int_0^{z/n} (z - na) f(a) da \right] + \tilde{r}_{t-1} \int_0^{z/n} \left(1 - \frac{na}{z}\right)^m f(a) da,
\]

and \(\hat{r}_0^* = 0\). Then, the proof follows by induction on \(t\), given the induction hypothesis \(\tilde{r}_t^* = n^{1-m} r_t^*\).
If \( t = 1 \), then, from (11) and (5),
\[
\tilde{r}_1(z) = \frac{n}{nm(z/n)^m} \left[ \frac{z}{n} - \int_{0}^{z/n} \left( \frac{z}{n} - a \right) f(a) \, da \right] = n^{1-m} r_1(z/n),
\]
which implies that \( \tilde{z}_1^* = nz_1^* \). Hence, \( \tilde{r}_1^* = n^{1-m} r_1^* \) and the induction hypothesis is true for \( t = 1 \). Therefore, assume that the induction hypothesis is true for \( t = i \), so that \( \tilde{r}_i^* = n^{1-m} r_i^* \). When \( t = i + 1 \), (11) and (5) imply
\[
\tilde{r}_{i+1}(z) = \frac{n}{nm(z/n)^m} \left[ \frac{z}{n} - \int_{0}^{z/n} \left( \frac{z}{n} - a \right) f(a) \, da \right]
+ \tilde{r}_i^* \int_{0}^{z/n} \left( 1 - \frac{a}{z/n} \right)^m f(a) \, da = n^{1-m} r_{i+1}(z/n).
\]
Thus, \( \tilde{z}_{i+1}^* = nz_{i+1}^* \), and \( \tilde{r}_{i+1}^* = n^{1-m} r_{i+1}^* \), which completes the proof. \( \square \)

One implication of Proposition 4 is that the optimal pricing policy is independent of the scale of the problem. To demonstrate this, suppose that, for a given problem scenario, the parameter \( \tilde{A}_t \) is replaced with \( \tilde{\tilde{A}}_t = n \tilde{A}_t \), then the optimal stocking factors change from \( z_t^* \) to \( \tilde{z}_t^* = nz_t^* \) for all \( t \). Moreover, the optimal stocking level changes from \( S_t^* \) to \( \tilde{S}_t^* = nS_t^* \) (because, from (7), \( \tilde{S}_t^* = (m \tilde{r}_T^*/c)^{1/(1-m)} \); and, from the proof of Proposition 4, \( \tilde{r}_T^* = n^{1-m} r_T^* \)). Thus, a change in problem parameters from \( A_t \) to \( \tilde{A}_t \) leaves the optimal prices unchanged in expectation: from (6),
\[
\tilde{p}_T^* = (nS_t^*/nz_t^*)(1-m) = p_T^*;
\]
\[
E[\tilde{p}_T^*] = E \left[ \frac{(nS_t^* - nA_T\tilde{p}_T^* - b)^+ / nz_{t-1}^*}{1-m} \right] = E[p_{T-1}^*]; \text{ etc.}
\]

This is in direct contrast to related results derived from dynamic pricing models in which demand is formulated as a Poisson process. In those models, if the demand rate and the initial inventory level both are scaled by a common factor, the optimal pricing policy changes as a result. This is because, in Poisson models, “scale” is not an independent parameter: an increase in the scale of demand results in a corresponding decrease in the demand coefficient of variation, which, in turn, affects the optimal price trajectory.

On a technical note, Proposition 4 is useful because it provides the flexibility to model any \( \tilde{A}_t \) of finite support \([0, w]\), simply by defining \( \tilde{A}_t = w \tilde{A}_t \), where \( \tilde{A}_t \) is a random variable defined over \([0, 1]\). We comment further on this observation, and the practical convenience it ensures, in Section 5.
The next proposition, together with its corollary, offer insight on how recourse affects the optimal stocking level, which is a one-time decision made at the beginning of the selling season, and the optimal expected profit. The insights come from comparing $S^*$ against the optimal $S$ for a price-setting newsvendor, which serves as the benchmark stocking level.

**Proposition 5.** The optimal stocking level, $S^*$, is at least as large as the benchmark quantity $S_B$, which is defined as the optimal stocking level for an otherwise equivalent decision scenario in which price is set only at the beginning of the selling season and then held fixed for the duration of the season.

*Proof.* First define $v_t(z)$ as follows:

$$v_t(z) = \frac{1}{z^m} \left( z - E \left[ \left( z - \sum_{j=1}^{t} A_j \right)^+ \right] \right). \quad \text{(12)}$$

Then, consider the following two lemmas, which we will establish in turn:

**Lemma 1.** $S_B = \left( m v_T^* / c \right)^{1/(1-m)}$, where $v_T^* = \max_z v_T(z)$.

**Lemma 2.** $r_T^* \geq v_T^*$.

Notice that if Lemmas 1 and 2 are true, then, from (7),

$$S^* = \left( m r_T^* / c \right)^{1/(1-m)} \geq \left( m v_T^* / c \right)^{1/(1-m)} = S_B.$$

Thus, to complete the proof of Proposition 5, it suffices to show that Lemmas 1 and 2 are true.

*Proof of Lemma 1.* If price is set only once and then held constant for the duration of the selling season, then, given $S$, the total revenue function for the season is

$$V_T(p|S) = p \left( S - E[\text{leftovers at the end of the season}] \right)$$

$$= p \left( S - E \left[ \left( S - p^{-b} \sum_{t=1}^{T} A_t \right)^+ \right] \right).$$

Correspondingly, the total profit function for the season is $V_T(p|S) - cS$. Defining $k = S/p^{-b}$ as the stocking factor for the price-setting newsvendor, and substituting it into the expression for $V_T(p|S)$, yields

$$V_T(p|S) = V_T \left( (k/S)^{1-m} | S \right)$$

$$= \left( \frac{k}{S} \right)^{1-m} \left( S - E \left[ \left( S - \frac{S}{k} \sum_{t=1}^{T} A_t \right)^+ \right] \right) = S^m v_T(k).$$
where \( v_T(k) \) is given by (12). Thus, \( S_B \) is the value of \( S \) that maximizes the concave function \( v_T^* S^m - cS \), where \( v_T^* = \text{arg max}_k v_T(k) \). That is, \( S_B = (m v_T^* / c) \frac{1}{1 - m} \), thereby establishing Lemma 1.

Proof of Lemma 2. This proof is by induction on \( t \), given the induction hypothesis \( r_t(z) \geq v_t(z) \) for all \( z \). If \( t = 1 \), then, from (5) and (12), \( r_1(z) = v_1(z) \). Therefore the induction hypothesis is true for \( t = 1 \). Assume, then, that the hypothesis is true for \( t = i \), so that \( r_i(z) \geq v_i(z) \), and consider the case \( t = i + 1 \). From (5),

\[
  r_{i+1}(z) = \frac{z - E[z - A_{i+1}]^+ \cdot (z - A_{i+1})^{+m}}{z^m}.
\]

(13)

However, by the definition of optimality, \( r_i^* \geq r_i(z) \) for all \( z \). Thus, \( r_i^* \geq r_i(z - A_{i+1}) \) for all realizations of \( A_{i+1} \leq z \). Consequently, for all \( A_{i+1} \leq z \), \( r_i^* \geq r_i(z - A_{i+1}) \geq v_i(z - A_{i+1}) \) by the induction hypothesis. Applying this inequality and (12) to (13) yields

\[
  r_{i+1}(z) \geq \frac{z - E[z - A_{i+1}]^+ + E[v_i(z - A_{i+1}) \cdot (z - A_{i+1})^{+m}]}{z^m}
\]

\[
  = \frac{z - E[(z - \sum_{j=1}^{i+1} A_j)^+]}{z^m} = v_{i+1}(z),
\]

which implies that the induction hypothesis is true for all \( t \). Let \( k_B = \text{arg max}_k v_T(k) \) so that \( v_T^* = v_T(k_B) \). Then, \( r_T^* \geq r_T(k_B) \geq v_T(k_B) = v_T^* \), where the first inequality follows by the definition of optimality and the second inequality follows because the induction hypothesis is true for all \( t \).

Intuitively, from an inventory-theory perspective, the flexibility to change selling price each period decreases the price-setting newsvendor’s cost of having leftovers in the following sense. If, coming into a new period, the actual stocking level exceeds the anticipated stocking level, a price change can stimulate demand, thereby establishing, in effect, a salvage market for the excess units. Thus, recourse in the form of flexibility to adjust prices allows the newsvendor to salvage leftovers that otherwise would not be salvaged. As a result, this recourse flexibility implicitly reduces the cost of having leftovers, which results in an increased optimal stocking level. Interestingly, similar intuition cannot be extended directly to the newsvendor’s initial pricing decision: the relationship between \( p_T^* \) and \( p_B \) (which is defined analogously
to $S_B$) is an ambiguous one. We investigate this ambiguity as part of our numerical study in Section 6.

With the following corollary, we demonstrate that the relative value of the newsvendor’s recourse flexibility is equivalent to the corresponding relative increase in stocking level, which itself can be expressed in succinct fashion.

**Corollary 1.** The relative expected value of being able to adjust prices dynamically over the course of a single selling season, which is defined as the ratio of the expected optimal profit of the dynamic pricing problem to the expected optimal profit of the price-setting newsvendor problem, can be expressed as follows:

$$ E[\text{value of recourse flexibility}] = \frac{r^*_T \cdot (S^*)^m - cS^*}{v^*_T S_B^m - cS_B} = \left(\frac{r^*_T}{v^*_T}\right)^b, $$

where $r^*_T$ is the maximum of (5), $v^*_T$ is the maximum of (12), and $b$ is the price elasticity of demand.

**Proof.** From (7), $S^* = (mr^*_T/c)^{1/(1-m)}$; and from Lemma 1,

$$ S_B = (mv^*_T/c)^{1/(1-m)}. $$

Thus, the corollary follows from elementary algebra. \qed

Notice that the expected value of recourse flexibility can be measured independent of $c$. This is a convenience that traces to (9), which indicates that the optimal expected profit for the selling season is log-linear in $c$. We further explore the expected value of recourse in Section 6.

We conclude this section by establishing the optimality of a single-price policy for the special case where demand each period is deterministic. We use this result in Section 6 to examine the loss of performance resulting from the employment of a commonly used heuristic that replaces random variables by their means in order to facilitate the computation of a pricing policy.

**Proposition 6.** For a given initial stocking level $S$, the pricing policy that is optimal for the deterministic case in which $d_t(p) = E[A_t] p^{-b}$ is a single-price policy: $p_t^* = p_d(S)$ for all $t$, where

$$ p_d(S) = \left(\sum_{t=1}^T E[A_t]/S\right)^{1-m}. $$
Proof. If each $A_t$ is replaced with $E[A_t]$ and is assumed to be deterministic, then the total revenue associated with the last $t$ periods of the selling season, given a beginning inventory of $I_t$, is

$$R_t(I_t) = \max_p \left\{ E[A_t] \frac{1}{p^{b-1}} + R_{t-1} \left( I_t - \frac{E[A_t]}{p^b} \right) \right\},$$

(14)

where $R_0(I) = 0$ for all $I$. Let $p_t(I_t)$ be the value of $p$ that satisfies (14). Then $p_t(I_t)$ denotes the optimal period-$t$ price for the case of deterministic demand, for a given $I_t$.

Given that demand is deterministic, the price in the last period of the selling season (period 1) should be set such that all remaining units are sold. That is,

$$p_1(I_1) = \left( \frac{E[A_1]}{I_1} \right)^{1/b},$$

which, from (14), implies that the period-1 contribution to revenue is

$$R_1(I_1) = p_1(I_1) I_1 = E[A_1]^{1/b} I_1^{1-1/b}.$$

Thus, assume as induction hypotheses that

$$p_t(I_t) = \left( \sum_{i=1}^{t} E[A_i]/I_t \right)^{1/b},$$

and that

$$R_t(I_t) = \left( \sum_{i=1}^{t} E[A_i] \right)^{1/b} I_t^{1-1/b}.$$

Then,

$$R_{t+1}(I_{t+1}) = \max_p \left\{ E[A_{t+1}] + \left( \sum_{i=1}^{t} E[A_i] \right)^{1/b} \left( p_{t+1} I_{t+1} - E[A_{t+1}] \right)^{1-1/b} \right\}. \frac{1}{p^{b-1}}.$$

After some algebra, this yields

$$p_{t+1}(I_{t+1}) = \left( \sum_{i=1}^{t+1} E[A_i]/I_{t+1} \right)^{1/b},$$

and, thus,

$$R_{t+1}(I_{t+1}) = \left( \sum_{i=1}^{t+1} E[A_i] \right)^{1/b} I_{t+1}^{1-1/b}.$$
thereby establishing that the induction hypotheses are true for all \( t \). Therefore, to complete the proof, it remains to be shown only that \( p_t(I_t) = p_T(S) \) for all \( t \). For all \( t \),

\[
I_{t-1} = I_t - E[A_t] p_t(I_t)^{-b} = I_t - \frac{E[A_t]}{\sum_{i=1}^{t-1} E[A_i]} I_t = \frac{\sum_{i=1}^{t-1} E[A_i]}{\sum_{i=1}^{t} E[A_i]} I_t.
\]

Therefore,

\[
p_{t-1}(I_{t-1}) = \left( \frac{\sum_{i=1}^{t-1} E[A_i]}{I_{t-1}} \right)^{1/b} = \left( \frac{\sum_{i=1}^{t} E[A_i]}{I_t} \right)^{1/b} = p_t(I_t)
\]

for all \( t \). Since \( I_T = S \) by definition, this implies that \( p_t(I_t) = p_T(I_T) = p_T(S) \) for all \( t \), which completes the proof.

In other words, if the demand is iso-elastic and deterministic, there is no need for price adjustments even if such opportunities exist. A single-price policy is generally not optimal, even in a deterministic setting, if the demand function is not iso-elastic (e.g., if demand is either linear or exponential).

## 5 Additional Properties For The Finite-Support Case

Notice from (5) that for \( t > 1 \), \( r_t(z) \) is a single-variable function having a “static” form (since, for a given \( t > 1 \), \( r^*_{t-1} \) is a constant that is strictly greater than zero). As a result, the procedure for solving the \( T \) optimization problems required to yield the complete set of \( \{z^*_t\} \) is no more difficult, computationally, than solving for \( z^*_2 \). Thus, the degree of difficulty associated with solving an instance of the dynamic pricing problem boils down to determining the shape of \( r_t(z) \) for a given value of \( r^*_{t-1} \). In implementing our algorithm for the numerical study presented in Section 6, we observed that \( r_t(z) \) typically is quasi-concave for a variety of continuous distributions used to characterize \( A_t \). Unfortunately, however, we have not been able to establish a proof to that effect; thus, as a worst-case scenario, determining \( z^*_t \) requires an exhaustive search over the feasible domain for \( z \). Fortunately, however, Propositions 2 and 3 limit the magnitude of the searches by establishing a lower bound for \( z^*_t \).

In this section, we further pare the space over which an exhaustive search is required for the special case in which \( A_t \) has finite support. (\( A_t \) is considered to have finite support if \( A_t \in [0, w] \) for \( w < \infty \).) In particular, we establish two key results for this case. The first result effectively establishes
that an exhaustive search is required only over the domain \([z^*_{t-1}, 1]\); if this
domain either is empty or does not yield \(z^*_t\), then \(z^*_t\) can be determined
uniquely by the first order condition \(dr_t(z)/dz = 0\). The second result estab-
lishes that no exhaustive search is required if \(A_t\) can be characterized by
the power distribution, \(F(a) = a^k\) for \(a \in [0, 1]\) and \(k > 0\). (Notice that
the power distribution is a subset of the Beta distribution; see, for example,
Bagnoli and Bergstrom (2001), for an introduction to the power distribution.)

**Proposition 7.** If \(A_t\) has finite support so that \(A_t \in [0, w]\), then \(r_t(z)\) is quasi-
concave for \(z \geq w\).

*Proof.* Assume that \(z \geq w\). This implies that \((z - A_t)^+ = (z - A_t)\); thus, from
(5),

\[
r_t(z) = \frac{1}{z^m} \left( E[A_t] + r^*_{t-1} \int_0^w (z - a)^m f(a) da \right).
\]

Consequently,

\[
\frac{dr_t(z)}{dz} = \frac{m r^*_{t-1}}{z^m} \int_0^w \frac{1}{(z - a)^{1-m}} f(a) da - \frac{m r_t(z)}{z} 
\]

\[
= \frac{m}{z} \left[ r^*_{t-1} \int_0^w \left( 1 + \frac{a}{z - a} \right)^{1-m} f(a) da - r_t(z) \right],
\]

which implies

\[
\left. \frac{d^2 r_t(z)}{dz^2} \right|_{dr_t(z)/dz = 0} = \frac{-m(1-m)r^*_{t-1}}{z^{1+m}} \int_0^w \frac{a f(a)}{(z - a)^{2-m}} da < 0.
\]

Therefore, \(dr_t(z)/dz\) can change sign at most once, from positive to negative,
over the region \(z \geq w\). In other words, \(r_t(z)\) is quasi-concave for \(z \geq w\). □

Proposition 7 is important because, used in conjunction with Propositions
3 and 4, it significantly reduces the domain over which an exhaustive search is
necessary. To demonstrate, suppose that the random component of demand
is given as \(\tilde{A}_t \in [0, w]\). Then, because of the scaling property established
as Proposition 4, the problem first can be rescaled so that \(\tilde{A}_t = w A_t\), where
\(A_t \in [0, 1]\). Correspondingly, from Proposition 7, \(r_t(z)\) is quasi-concave for
\(z \geq 1\). Moreover, from Proposition 3, \(z^*_t \geq z^*_t\). Thus, for a given value
of \(z^*_t\), if \(z^*_t \geq 1\), then \(z^*_{t+1}\) can be found simply as the unique solution to
\(dr_{t+1}(z)/dz\). In other words, once it is determined that \(z^*_t \geq 1\) for some \(t\),
then for all \(j > 0\), \(z^*_{t+j}\) can be computed directly and efficiently from its first-
order condition. (Our numerical investigation suggests that, typically, \(z^*_t \geq 1\)
even for relatively low values of $t$.) If $z_t^* < 1$, then an exhaustive search for $z_t^{*+1}$ is required, but only over the region $[z_t^*, 1]$; the best candidate from this region then can be compared with the best candidate from the region $z \geq 1$, which can be determined efficiently since $r_{t+1}(z)$ is guaranteed to be well-behaved for $z \geq 1$. Finally, once $z_t^*$ is determined, $\tilde{z}_t^*$, the optimal period-$t$ stocking factor for the original problem (in which $\tilde{A}_t \in [0, w]$ is given) can be recovered, from Proposition 4, as $\tilde{z}_t^* = w z_t^*$.

We conclude this section with Proposition 8, which establishes that, for the special class of Beta distributions in which one of the two parameters is identically equal to 1, $r_t(z)$ is quasi-concave overall $z$; thus, $z_t^*$ can be found directly from its first-order condition for all $t$.

**Proposition 8.** If $A_t$ has a power distribution so that $F(a) = a^k$ for $a \in [0, 1]$ and $k > 0$, then $r_t(z)$ is quasi-concave for all $z$.

**Proof.** Assume that $F(a) = a^k$ for $a \in [0, 1]$ and $k > 0$. This implies that $f(a) = k a^{k-1}$ for $a \in [0, 1]$; and $f(a) = 0$ otherwise. Thus,

$$E \left[ \left( 1 - \frac{A_t}{z} \right)^m \right] = \int_0^{\min\{z, 1\}} k \left( 1 - \frac{a}{z} \right)^m a^{k-1} da = \int_0^{\min\{1, 1/z\}} k z^k (1 - v)^m v^{k-1} dv,$$

which, from (5), implies

$$r_t(z) = \begin{cases} z^{1-m} \left[ 1 - \int_0^1 k z^k (1 - v)^m v^{k-1} dv \right] + r_{t-1}^* \int_0^1 k z^k (1 - v)^m v^{k-1} dv & \text{if } z < 1 \\ \frac{E[A_t]}{z^m} + r_{t-1}^* \int_0^{1/z} k z^k (1 - v)^m v^{k-1} dv & \text{if } z \geq 1. \end{cases}$$

Or,

$$r_t(z) = \begin{cases} z^{1-m} \left[ 1 - \frac{z^k}{k+1} \right] + r_{t-1}^* \frac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(k+m+1)} z^k & \text{if } z < 1 \\ \frac{k/(k+1)}{z^m} + r_{t-1}^* \int_0^{1/z} k z^k (1 - v)^m v^{k-1} dv & \text{if } z \geq 1. \end{cases}$$

where $\Gamma$ denotes the Gamma function. Notice that $r_t(z)$ is continuous. More-
over,

\[
\frac{dr_t(z)}{dz} = \begin{cases} 
\frac{1}{z^m} \left[ (1 - m) - \frac{(1 - m + k)z^k}{k + 1} \right] & \text{if } z < 1 \\
+kr^*_t \left\{ \frac{\Gamma(m + 1)\Gamma(k + 1)}{\Gamma(k + m + 1)} \right\} z^{k-1} & \\
kr^*_t \left( k^{k-1} \int_0^{1/z} (1 - v)^m v^{k-1} dv - \frac{1}{z} \left( \frac{1 - 1}{z} \right)^m \right) & \\
+ \frac{-mk}{(k + 1)} z^{m+1} & \text{if } z \geq 1.
\end{cases}
\]

Thus, \( r_t(z) \) also is differentiable.

Since \( r_t(z) \) is continuous and differentiable, and since \( r_t(z) \) is quasi-concave for \( z \geq 1 \) (by Proposition 7), it suffices to show that \( r_t(z) \) is quasi-concave for \( z < 1 \) to complete the proof. For \( z < 1 \),

\[
\frac{d^2 r_t(z)}{dz^2} = \frac{-m}{z} \frac{dr_t(z)}{dz} < 0,
\]

which follows directly from (16). Likewise, if \( k > 1 - m \), then

\[
\frac{d^2 r_t(z)}{dz^2} \bigg|_{\frac{dr_t(z)}{dz} = 0} < 0,
\]

because (15) applied (16) yields

\[
\frac{d^2 r_t(z)}{dz^2} \bigg|_{\frac{dr_t(z)}{dz} = 0} = \frac{1}{(k + 1)z^m} \left[ -k(k + 1 - m)z^{k-1} \right.
\]
\[
\left. + \frac{k - 1 + m}{z} \left( (1 - m + k)z^k - (1 - m)(k + 1) \right) \right] < -(1 - m) \frac{k + 1 - m}{k + 1} z^{k-1-m} < 0.
\]
Thus,

\[ \frac{d^2 r_t(z)}{dz^2} \bigg|_{dr_t(z)/dz = 0} < 0 \]

is true for both cases, which implies that \( r_t(z) \) is quasi-concave for \( z < 1 \). This completes the proof. \( \square \)

### 6 Numerical Results

In this section, we develop additional insights to complement those established in earlier sections by implementing our solution procedure for a wide variety of probability distributions and problem parameters. Specifically, we report the findings of three experiments designed to develop some intuition with regard to the following questions:

1. What is the “cost” of using well-established heuristics for solving the dynamic-pricing problem in lieu of computing the optimal solution?

2. How does the recourse flexibility that is inherent to the dynamic-pricing problem affect the optimal price chosen to start the selling season (when compared to the price-setting newsvendor benchmark case); and what is the magnitude of the value of that recourse?

3. How does the optimal price sequence behave over time, in expectation?

We also explore how the answers to these questions are affected by changes in demand elasticity, coefficient of variation, and season length.

In the results presented here, the Gamma distribution \( \Gamma(\alpha, \beta) \) is used for the distribution of \( A_t \). The Gamma distribution is a two-parameter distribution, whose mean is \( \alpha \beta \), and variance is \( \alpha \beta^2 \).

Figure 1 plots \( r_t^* \), which, in light of Proposition 1, is a surrogate for optimal expected revenue generated for a fixed supply over a single season, as a function of \( T \). In this example, \( \alpha \beta = 10 \) and the per-period coefficient of variation (CV = \( \alpha^{-1/2} \)) takes on values of CV = 2.0, 1.5, and 0.5. As expected, optimal expected season revenue increases as a function of the length of the season and decreases as a function of the per-period coefficient of variation of demand.
6.1 Loss of Performance Due to Static Pricing

As indicated in the introduction, optimal dynamic pricing policies typically are difficult to compute without first specifying a structural relationship between the demand process and price. As a consequence, heuristics often are developed and applied. In this section, we use our model to investigate how financial performance might suffer if one such heuristic is used in lieu of optimally solving the dynamic problem, for a given (fixed) amount of inventory.

The deterministic heuristic, in which random variables are replaced by their means, is commonly employed to compute approximately optimal prices in dynamic pricing problems. From Proposition 6, we know that the deterministic heuristic yields a single-price policy. Accordingly, we suggest that a better price to use is $p_B(S)$, the price set by an optimizing price-setting newsvendor whose initial stock is $S$. From Proposition 5, $p_B(S) = (k_B/S)^{1/b}$, where $k_B$ is the price-setting newsvendor’s optimal safety factor. (See the proof of Proposition 5 for details.) Since this price is optimal in the original stochastic setting of the problem with the added stipulation that price can be set only once, it dominates all other single-price policies, including the deterministic heuristic. Moreover, it is effectively as easy to implement as the deterministic heuristic. Thus, to better understand the performance loss resulting from applying a heuristic to the dynamic pricing problem, we compare $v^*_T S^m$, the expected total revenue associated with trying to sell a fixed supply $S$ over a finite horizon $T$ using the single-price policy $p_B(S)$, to $r^*_T S^m$, the expected total revenue associated with trying to sell the same $S$ units over the same time $T$ using the optimal pricing policy $p^*_T$. That is, we compute the ratio $r^*_T / v^*_T$, which is independent of $S$.

Figure 2(a) shows graphs of $r^*_T / v^*_T$ as a function of $T$ for $CV = 2.0, 1.5, \ldots$. 

![Figure 1: $r^*_T$.](image-url)
We find that the percentage loss resulting from using the single-price policy $p_B(S)$ in the dynamic pricing problem is as high as 5%. Notice that the performance loss is not monotone in CV: when the season is short ($T \leq 3$), performance losses are greater for small values of CV ($CV = 0.5$) than they are for larger values of CV. On the other hand, when the season is longer ($T \geq 7$), performance losses are smallest for $CV = 0.5$ and largest for $CV = 1.0$.

Figure 2(b) shows graphs of $r^*_T / v^*_T$ as a function of $T$ for three values of $b$ ($b = 1.5, 2.0, and 2.5$), the elasticity parameter. We see that for each value of $T$, performance losses are not monotonically increasing in $b$. In particular, for each $T$, the greatest performance loss occurs when $b = 2.0$.

(a) Varying CV.  
(b) Varying $b$.

Figure 2: Revenue performance loss comparisons.

### 6.2 Value and Effect of Recourse

Because it includes only one stocking opportunity, the dynamic pricing problem studied in this paper can be thought of as a price-setting newsvendor problem with recourse. The recourse comes in the form of flexibility to adjust prices dynamically as demand is observed. Accordingly, it is interesting to explore how the amount of recourse (i.e., the number of price changes associated with a given season’s worth of demand) affects a price-setting newsvendor’s optimal policy and corresponding optimal profit.

Recall that $S_B$, $p_B$, $k_B$, and $v^*_T$ denote the price-setting newsvendor’s optimal stocking quantity, optimal selling price, optimal stocking factor, and optimal revenue factor, respectively; and that these quantities serve as the benchmarks for comparison. Recall also that Proposition 5 and its corollary
already establish that $S^* \geq S_B$, and that the relative value of recourse can be measured simply as the ratio of $S^*$ to $S_B$. Here, we further investigate that ratio $S^*/S_B = (r_T^* / v_T^*)^b$, to develop keener insight into the relative value that pricing recourse provides to the price-setting newsvendor as well as how that value depends on the problem primitives. In addition, we explore the ratio

$$\frac{p_T^*}{p_B} = \left( \frac{z_T^* S_B}{k_B S^*} \right)^{1-m} = \frac{v_T^*}{r_T^*} \left( \frac{z_T^* \\kappa S_B}{k_B} \right)^{1-m}$$

(17)

to develop an understanding of how a newsvendor’s optimal pricing strategy is affected by the amount of recourse available.

In this context, $T$ denotes the number of price adjustments that a newsvendor makes during the course of a given season. Accordingly, in Figures 3 and 4, we assume that the uncertainty associated with seasonal demand, $\sum_{t=1}^T A_t$, has a Gamma distribution $\Gamma(\alpha, \beta)$, so that $A_t \sim \Gamma(\alpha/T, \beta)$, and we let $T = 1, 2, \ldots, 12$. Figure 3 shows that $100 (r_T^* / v_T^*)^b - 1$, the percentage gain in profit resulting from pricing recourse, is increasing in $T$ for all values of $\text{CV} = \alpha^{-1/2}$ and $b$. Therefore, a newsvendor’s optimal profit increases as the number of price adjustments over the course of a season increases. However, the marginal return of each additional price change diminishes. Thus, when $T$ is large, the marginal benefit of making an additional price change may not make up for the cost of doing so.

Figure 4 shows that $100(p_T^*/p_B - 1)$, the percentage change in the (initial) optimal price resulting from pricing recourse, seems to depend primarily on the uncertainty in demand. Interestingly, if the demand coefficient of variation is either relatively high or relatively low, then the newsvendor with recourse tends to start the season with a lower price than a newsvendor without recourse (i.e., $100(p_T^*/p_B - 1) < 0$). Otherwise, the newsvendor with recourse tends to start the season with a higher price than the newsvendor without recourse.
Figure 3: Expected value of recourse flexibility (as a percentage gain) with a fixed distribution of seasonal demand.
Figure 4: The impact of recourse on the newsvendor's choice of price (as a percentage change).
6.3 Optimal Price Sequence

From (6), the actual period-\(t\) optimal price depends on \(I_t\), the observed level of inventory available at the beginning of the period (which depends on the period-\((t+1)\) realization of demand). However, the amount of inventory available at the beginning of period \(t\) is equivalent to the number of leftovers from period \(t + 1\), which is a function of the period-\((t + 1)\) optimal price:

\[
I_t = \frac{I_{t+1}}{z^*_{t+1}} (z^*_{t+1} - A_t) + \frac{(z^*_{t+1} - A_t)^+}{(p^*_{t+1})^{1/(1-m)}}.
\]

Therefore, in expectation, the ratio of prices from one period to the next throughout the selling season is a straightforward comparison that is independent of the stocking level:

\[
E \left[ \frac{p^*_t}{p^*_{t+1}} \right] = E \left[ \frac{(z^*_t/I_t)^{1-m}}{p^*_{t+1}} \right] = E \left[ \left( \frac{z^*_t}{(p^*_{t+1})^{1/(1-m)}} \right)^{1-m} I_t \right] = \frac{(z^*_t)^{1-m}}{E \left[ \left( (z^*_{t+1} - A_t)^+ \right)^{1-m} \right]},
\]

for all \(t\). Hence, (18) provides a convenient expression for testing the behavior of optimal prices over time for a fixed season length. When (18) yields a ratio that is greater than 1, the indication is an expected price increase from period \(t\) to \(t + 1\); and when (18) yields a ratio that is less than 1, the indication is an expected price decrease from period \(t\) to \(t + 1\).

Figure 5 shows typical results. In this representative case, we again use the Gamma distribution \(\Gamma(\alpha, \beta)\) as the distribution of \(A_t\); and we consider a 12-period horizon. For various values of \(CV = \alpha^{-1/2} \) and \(b\), we find that \(p^*_t\) typically increases during earlier periods of the season (i.e., when there are many periods remaining), and tends to decrease as the finite horizon draws to an end.
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(a) Varying CV.

(b) Varying $b$.

Figure 5: $E \left[ \frac{p_t^*}{p_{t+1}^*} \right]$. 
7 Conclusion

We have considered the problem of determining the optimal prices to set in each period of a finite selling season when the stocking level can be chosen only at the beginning of the season and demand is modeled as an iso-elastic function of price with a multiplicative uncertainty term characterized by any given probability distribution. In general, the period-$t$ optimal price depends on the inventory state of the system. However, we show that this dynamic pricing problem can be reformulated as a dynamic stocking-factor problem that is independent of the inventory state of the system. As a result, the problem can be solved with relative ease using an iterative routine in which a stand-alone revenue function is maximized for $T$ separate specifications that differ only by the magnitude of a single coefficient term. In effect, the solution procedure requires solving $T$ single-period problems, each involving a time-dependent “revenue parameter” that is exogenous to the particular iteration of the solution routine.

We further show that the first iteration of the solution routine (which corresponds to the trivial case in which the “revenue parameter” is identically equal to zero) is quasi-concave under the robust condition of an increasing generalized hazard rate for the distribution of the $A_t$. Thus, the final-period optimal stocking factor can be found directly from its first-order condition under quite general circumstances. And, for cases in which the uncertainty in demand is bounded by finite support $w$ (so that $A_t \in [0, w]$), we establish sufficient conditions indicating when the optimal stocking factor for any iteration can be determined directly from its first-order condition, namely for situations in which it is known that the optimal stocking factor is no less than $w$ or in which $A_t$ can be characterized by a power distribution. We also demonstrate that the optimal solution is independent of the scale of the problem; hence, large problems may be rescaled to a more convenient size before implementing the solution routine.

For the case in which the $A_t$ are identically distributed, we show that the optimal stocking factors are monotone increasing as a function of the number of periods remaining in the finite horizon. This result thus captures the essence of the relationship between the dynamic pricing problem and dynamic inventory theory, thereby inspiring an interpretation of the dynamic pricing problem as a price-setting newsvendor problem with recourse. This resulting interpretation is useful not only because it yields insights into the optimal solution, but also because it leads to additional insights into how pricing recourse affects the actions and profits of a price-setting newsvendor. Consequently, we find that a price-setting newsvendor will choose a
higher stocking level when recourse is available. However, the effect that the recourse flexibility has on the newsvendor’s corresponding selling price depends on the level of uncertainty inherent in the problem: a relatively lower or relatively higher demand coefficient of variation tends to drive the newsvendor with recourse to choose a lower initial price than the newsvendor without recourse, but intermediate demand coefficients of variation tend to drive the newsvendor with recourse to choose a higher initial price. Finally, the relative value of the recourse flexibility can be expressed simply as the ratio of the optimal stocking level for the price-setting newsvendor with recourse to the optimal stocking level for the price-setting newsvendor without recourse.

Although we have assumed that the $A_t$ are identically distributed, this assumption is not essential in the analysis. If, instead, the $A_t$ were non-stationary, than all of the propositions except for Proposition 2 would continue to hold. Thus, the general solution procedure and the insights developed in this paper would still apply. The only effect of having non-stationary demand, therefore, is a technical one: if demand is not stationary, then the monotone relationship between the optimal stocking factors is not guaranteed. Hence, in the worst case, an exhaustive search would be necessary at each iteration of the solution procedure. Note, however, if the demand process is not stationary, then one possible approach would be to redefine the “periods” of the finite horizon so that expected demand in each period is roughly the same. This approach leads to periods that are of unequal length (in terms of time), but it does not affect our solution technique.

A more interesting extension, perhaps, is the development of a model in which the stocking level can be replenished at representative intervals, though not as frequently as price can be adjusted. Extrapolating the description of our model, this situation would be characterized as one in which there were multiple selling seasons (say, $N$), each composed of $T$ periods (i.e., opportunities) for setting the price. Solving this extension seems to require more than a trivial adaptation of our solution procedure because the resulting optimal safety factor decisions appear not, in general, to be independent of the inventory state of the system as they are when there is only a single stocking opportunity.

To provide insight into this phenomenon, consider a 2-season model. At the end of the first season, the decision maker is faced with a single-season, dynamic pricing problem like the one studied here, except that some number of units, say, $x$, will be available before the stocking decision is settled upon. As to be expected, the resulting implication is for the decision maker to replenish stock only if $x$ is small; if $x$ is relatively large, then it behooves
the decision maker not to replenish and to operate in the second season with \( x \) units. As a result, the \( T \)-period optimal value function associated with the second of the two selling season turns out to be concave in \( x \). This function, in turn, effectively represents a terminal-value salvage function for leftovers associated with the last period of the first selling season. Consequently, the decision problem at the beginning of the first selling season is similar to the single-season dynamic pricing problem studied here except that there is a concave end-of-season salvage function associated with leftovers. Unfortunately, this embellishment to the dynamic pricing problem then appears to lead to optimal stocking factors for the first selling season that depend on the inventory state of the decision.

Given that the dynamic pricing problem studied in this paper can be reinterpreted as a price-setting newsvendor problem with (pricing) recourse, the multiple-season extension introduced in the previous paragraphs can thus be thought of as a dynamic pricing and inventory model with (pricing) recourse. From a theoretical point of view, this extension is potentially appealing because it stands to continue the interpretive analogy between dynamic pricing theory and dynamic inventory theory. Indeed, we conjecture that the reflective symmetries between the two theories will continue to exist. Consequently, just as the newsvendor model provides the fundamental building block for dynamic inventory theory, and just as the price-setting newsvendor model offers a stepping stone for studying joint (periodic) inventory and pricing decisions, a reinterpretation of the dynamic-pricing problem as a price-setting newsvendor model with recourse has the potential to stimulate the merging of insights from dynamic pricing and dynamic inventory models. In turn, this could lead to a more basic decision-analytic framework for understanding the coordination of supply and demand in an uncertain world.

We conclude with a final note on the iso-elastic demand function. Because of its tractability, parsimony, robustness, and rich history in economic modeling, we feel it serves as a promising candidate for incorporating the effects of endogenous demand into more traditional operations management models, thereby enhancing the rigorous study of higher-level managerial decision making.

References

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