A Technical Analysis of the Quality of Real Estate Services

We have just seen that a firm acting optimally will select that level of rent, and therefore of occupancy, for which marginal operating expense is just equal to marginal effective gross income. The firm's resulting net operating income (*NOI*) was shown to be rent minus average operating expense at the optimal occupancy level, multiplied by the number of occupied units. This outcome was the best that the firm, facing particular demand and marginal operating expense curves, could attain. Of course, when we considered the relationship between the quality of real estate services and the level of rent, we saw that the firm potentially realizes many different optimal *NOI*s, one for each level of service quality. The key question therefore becomes which of the potentially optimal *NOI*s is *truly* optimal.

To identify this true optimal *NOI*, we must first specify the way in which rent responds to changes in the quality of real estate services. Second, we must identify precisely how average operating costs respond to changes in the quality of real estate services. Finally, after incorporating some basic mathematical equations, we can identify the truly optimal *NOI*. We can begin by reviewing the analytical framework of the previous article, while providing an extra measure of technical detail along the way.

Seeking the Optimum

In the upper right quadrant of Figure 1, rent, or average effective gross income (*AEGI*), is represented as a downward-sloping straight line. The relationship is shown as linear because the relationship between *EGI* and the proportion of units that are occupied is assumed to first increase and then decrease, in a manner represented graphically as a *parabola*, as the occupancy level increases. This result can be expressed mathematically as:

$$EGI = \alpha U - \beta U^2$$

an equation in which both α and β are greater than zero and *U* is the number of occupied units. Dividing both sides by the number of occupied units, *U*, results in the rent, or *AEGI*, function's being a straight line:

$$R = AEGI = \alpha - \beta U$$

an equation in which α is the vertical intercept and β is the slope, or steepness, of this demand "curve." Marginal effective gross income (*MEGI*), which has the same intercept as *AEGI* but a slope that is twice as steep, is the change in *EGI* attributable to a one-unit change in the number of occupied units. *MEGI* is computed as the *first derivative* of the *EGI* function:

$$\frac{dEGI}{dU} = \alpha - 2\beta U$$

Operating expenses are assumed to increase along a straight line as the number of occupied units increases:

$$OE = a + bU$$

an equation in which both *a* and *b* are, again, assumed to be positive. Marginal operating expense, or the change in total operating expenses attributable to a one-unit change in occupancy, is computed as the following constant:

$$\frac{dOE}{dU} = b$$

The optimal occupancy is the level for which marginal operating expenses are just equal to marginal effective gross income. This relationship implies that

$$\alpha - 2\beta U^* = b$$

an equation for which U^* is the level of occupancy that maximizes *NOI*. Solving for U^* allows us to find the number of occupied units that maximizes *NOI*:

$$U^* = \frac{\alpha - b}{2\beta}.$$

Substituting the optimal occupancy into the rent (demand) function, we find the optimal rent:

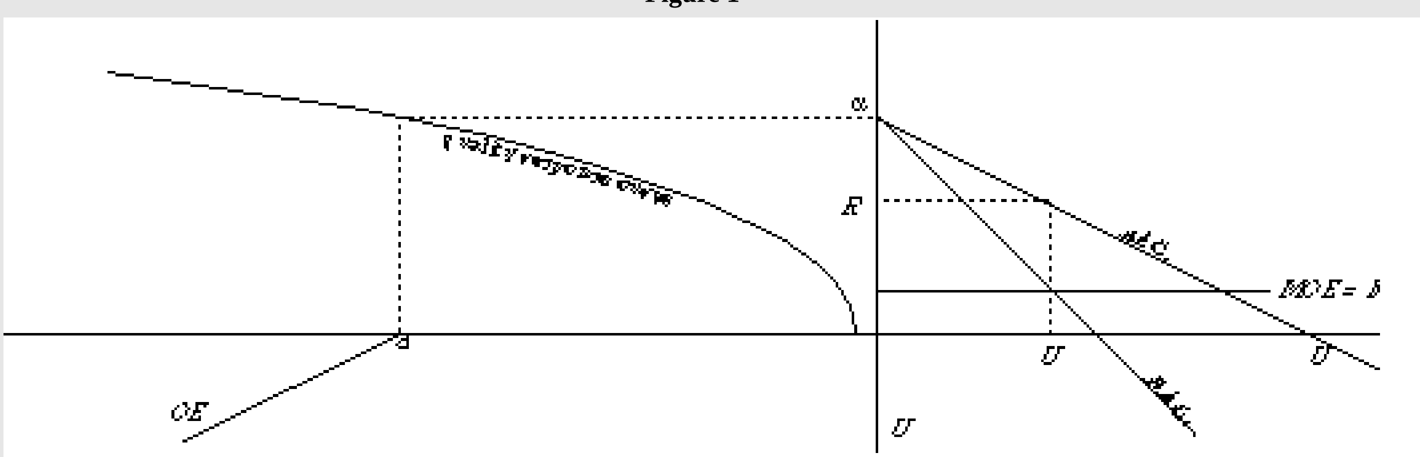
$$R^* = \alpha - \frac{\alpha - b}{2}.$$

Note that rent is one of the primary decision variables in this model. Now that we have reestablished the way in which optimal occupancy and rent are determined, let us turn to the focus of this technical discussion: the effect of changes in the quality of real estate services on *NOI*.

Total Operating Expenses & Demand

Total operating expenses, *OE*, are shown in the lower left quadrant of Figure 1.

Figure 1



A more conventional way to represent operating expenses graphically would be as a positive, increasing function in a quadrant for which the number of units appears on the horizontal axis and price is shown on the vertical axis. In the lower left quadrant of Figure 1, this conventional picture has been inverted, such that units appear on the vertical axis and price is on the horizontal axis. The intercept of the operating expense function, denoted a, can be thought of as the level of operating expenses incurred with 100% vacancy (0% occupancy).

Now, recall the relationship we have assumed between rent and operating expenses: that the intercept of the rent, or *AEGI*, function increases at a decreasing rate as the intercept of the operating expense function, a, increases. The *quality response curve* in the upper left quadrant of Figure 1 shows the intercept of the demand curve, α, that corresponds to the intercept of the operating expense function, a. The fact that the quality response curve increases at a decreasing rate with increases in a can be represented symbolically as:

$$\alpha = f(a)$$

$$\text{where } f'(a) > 0 \text{ and } f''(a) < 0$$

The meaning of the first statement is that the height of the rent function, α, depends on the height of the operating expense function (in Figure 1, the horizontal intercept a). The first portion of the latter statement, $f'(a) > 0$, simply means that as a increases, α increases. The second portion, $f''(a) < 0$, means that the size of the increase in α gets smaller with each successive increase in a. (Using mathematical terminology, we would say that the first derivative is positive and the second derivative is negative.) In summary, the height of the demand curve, *AEGI*, increases at a decreasing rate with increases in the height of the operating expenses function.

Average and Total Operating Expenses

Recall the definition of net operating income: rent per unit minus average operating expense per unit, multiplied by the number of units occupied. As discussed in the previous article, average operating expenses increase as service quality improves; this situation is shown graphically as an increase in the height of the operating expense curve, a. Yet while we can observe average operating expenses, how do we know exactly what the level of quality, α, is?

Figure 2 demonstrates how, given a particular average operating expense curve, we can detect the level of quality, and consequently the height of the *AEGI* (or demand) curve associated with this level of quality. In that figure, the firm faces average operating expense curve $AOE(a^\dagger)$. The meaning of this notation is that average operating expenses depend on the horizontal intercept of the operating expense curve ($a^\dagger$, a magnitude that is yet to be identified). Let us randomly select some level of occupancy in Figure 2, perhaps four units. At this occupancy level, we know what average operating expenses are, and we also know what marginal operating expenses are. Now, recall the formula for computing operating expenses:

$$OE = a + bU$$

Dividing each side of this equation by the number of occupied units, we can compute average operating expenses, or operating expenses per unit, as

$$\frac{OE}{U} = \frac{a}{U} + b$$

Rearranging algebraically, we can see that a divided by the number of units must be equal to

$$\frac{a}{U} = \frac{OE}{U} - b$$

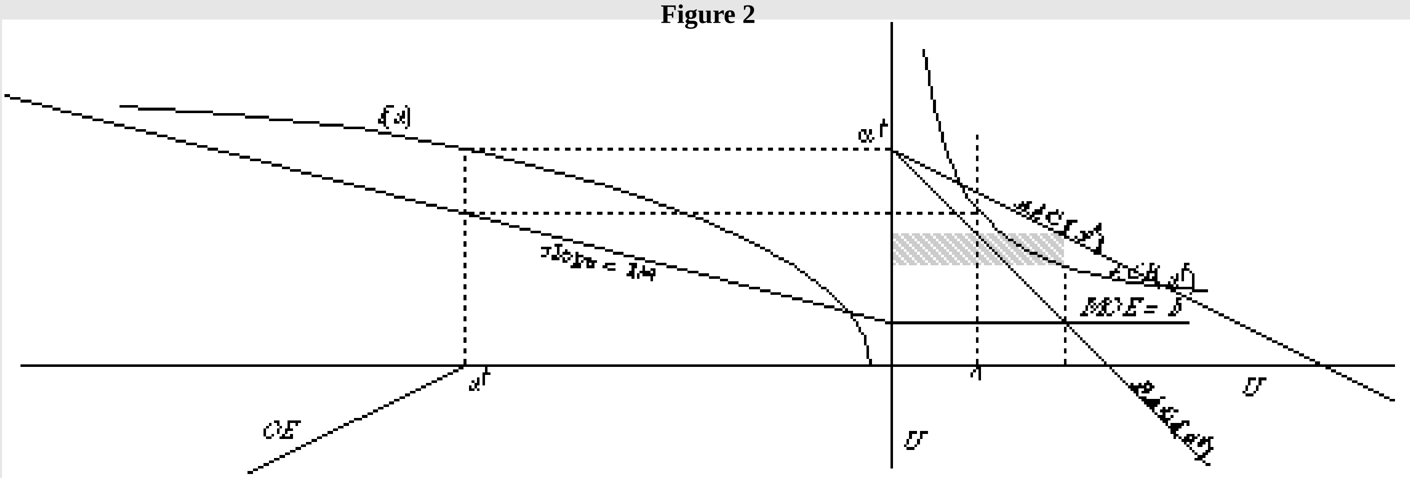
If $U = 4$, then

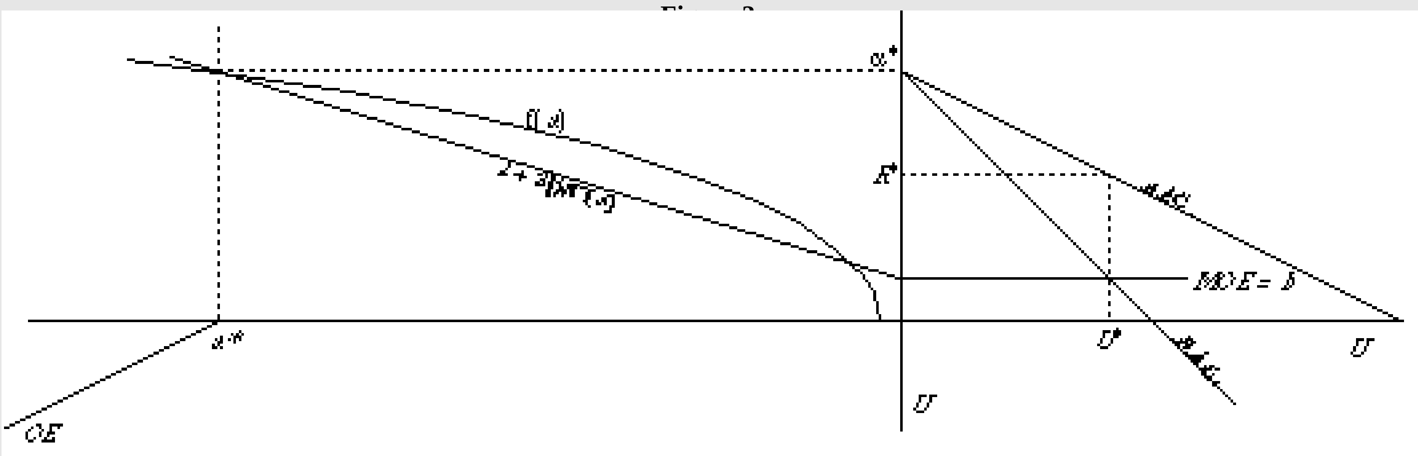
$$\frac{a^\dagger}{4} = AOE^\dagger - b$$

In Figure 2, we can identify $a^\dagger$ as four times the difference between average and marginal operating expenses at a four-unit level of occupancy.

Another trick to identify $a^\dagger$ is to draw a line that emerges from the vertical axis at b, with a slope equal to one divided by the number of units at which *AEGI* is measured. The result is shown in Figure 2 as the dark line with a slope equal to one-fourth. Given average operating expenses that prevail at four units of occupancy, we can read horizontally from the height of $AOE(a^\dagger)$ to the point where this new line is crossed; we can verify visually that the resulting horizontal distance is $a^\dagger$. We can reach this same result algebraically

Figure 2





by rearranging terms in the previous equation to show

$$AOE^{\dagger} = \frac{1}{4}a + b$$

Therefore, when an increase in total operating expenses causes average operating expenses to increase, we can observe this average amount at four units in order to determine the level of quality, a. Of course, we could have picked any number of units – for example, eight – at which to observe average operating expense. The slope of our line emerging from *b* would then be one-eighth, rather than one-fourth.

Now that we have identified $a^{\dagger}$ in Figure 2, we can use the quality shift function, $f(a)$, to identify the intercept of the rent function. We can then select the optimal level of rent, and the optimal level of occupancy, for a specified average and marginal effective gross income, and then calculate the maximum *NOI* associated with this level of service quality. This *NOI* level is represented graphically as the shaded rectangle in Figure 2.

If average operating expenses were higher at every level of occupancy, we would observe a new *AOE* curve corresponding to a higher level of quality of real estate services. But an increase in the level of quality, and therefore in observed operating costs, does not necessarily decrease total profit. Although there is a decline in net operating income *per unit*, the overall loss in *NOI* on previously occupied units (for which higher rents had previously been paid) is less than the increase to *NOI* that the manager realizes from the additional occupancy of units.

The Maximum Optimal *NOI*

The firm attains its optimal *NOI* when it sets rent per unit equal to the value indicated along the demand function, at the level of occupancy for which marginal operating expense (*MOE*) is just equal to marginal effective gross income (*MEGI*). Of course, we have established that changes in the quality of real estate services affect both average operating expenses and the level of demand, such that there is a unique optimal *NOI* for every level of quality. We can identify the maximum from among all of the potential optimum *NOIs* by working with the definition of net operating income:

$$NOI^{\dagger} = EGI - OE$$

or, substituting from our definitions of *EGI* and *OE*:

$$NOI^{\dagger} = \alpha U^{\dagger} - \beta U^{\dagger 2} - [a + bU^{\dagger}]$$

Next, substituting the optimal occupancy, U^*, and the link between the intercepts of the rent and expense functions, $\alpha = f(a)$, into the *NOI* function and expanding produces the following relationship:

$$NOI^{\dagger} = \frac{f^2(a)}{4\beta} - \frac{2bf(a)}{4\beta} - a - \frac{b^2}{4\beta}$$

This equation describes the *NOI*s that are optimal in light of each possible value for the intercept, a, of the operating expense function. However, at this stage of our analysis we still do not know how to choose the optimal level, a^*, of the operating expense function. In order to determine a^*, we must discover how NOI^* changes as a changes, and then set that level of change equal to zero. In this way (again, computing a first derivative), we can find the maximum from among all the optimal *NOI*s:

$$\frac{dNOI^{\dagger}}{da} = \frac{f(a)f'(a)}{2\beta} - \frac{bf'(a)}{2\beta} - 1$$

We simplify this relationship algebraically, and set both sides of the equation equal to zero such that:

$$f(a)f'(a) - bf'(a) = 2\beta$$

Rearranging so that we are able to show the optimal intercepts of both the rent function and the operating expense function produces

$$f(a) = b + \frac{2\beta}{f'(a)}.$$

Of course, the left-hand side of this equation is simply our quality response curve, $f(a)$. Figure 3 provides a graphical representation of the right-hand side of this equation. Note the following problem: the quality response curve and the curve representing the right-hand side of the above equation intersect at two different points. One of these points generates the *minimum NOI*, while the other generates the *maximum NOI*. The *OE* curve and rent functions shown in Figure 3 are the ones that are associated with the maximum *NOI*. The optimal level of quality is a^*, and the optimal height of the rent function is α^*. Corresponding to these values, the optimal level of occupancy for the firm is U^*, and the optimal rent to charge per unit is R^*. These particular values of occupancy and rent will yield the *maximum* optimal net operating income to the firm from among all the potential optimal *NOI*s. ■