Chapter 18

Models of Credit Risk

To this point, our models of bond pricing assumed that bonds have no risk of default. Therefore, these models are most applicable to valuing default-free bonds issued by a federal government, which would include Treasury bills, notes, and bonds. However, many debt instruments, including corporate bonds, municipal bonds, and bank loans, have default or "credit" risk. Valuing defaultable debt requires an extended modeling approach. We now consider the two primary methods for modeling default risk. The first, suggested in the seminal option pricing paper of Fischer Black and Myron Scholes (Black and Scholes 1973) and developed by Robert Merton (Merton 1974), Francis Longstaff and Eduardo Schwartz (Longstaff and Schwartz 1995), and others, is called the "structural" approach. This method values a firm's debt as an explicit function of the value of the firm's assets and its capital structure.

The second "reduced form" approach more simply assumes default is Poisson process with a possibly time varying default intensity and default recovery rate.

1Default can be avoided on government bonds promising a nominal (currency-valued) payment if the government (or its central bank) has the power to print currency. If a federal government gives up this power, such as governments whose official currency is the Euro that is supplied by the European Central Bank, default on government debt becomes a possibility.
This method views the exogenously specified default process as the “reduced-form” of a more complicated and complex model of a firm’s assets and capital structure. Examples of this approach include work by Robert Jarrow, David Lando, and Stuart Turnbull (Jarrow, Lando, and Turnbull 1997), Dilip Madan and Haluk Unal (Madan and Unal 1998), and Darrell Duffie and Kenneth Singleton (Duffie and Singleton 1999). This chapter provides an introduction to the main features of these two methods of modeling default.

18.1 The Structural Approach

This section considers a model similar to that by Robert Merton (Merton 1974). It specifies the assets, debt, and shareholders’ equity of a particular firm. Let $A(t)$ denote the date $t$ value of a firm’s assets. The firm is assumed to have a very simple capital structure. In addition to shareholders’ equity, it has issued a single zero-coupon bond that promises to pay an amount $B$ at date $T > t$. Also let $\tau \equiv T - t$ be the time until this debt matures. The firm is assumed to pay dividends to its shareholders at the continuous rate $\delta A(t) dt$, where $\delta$ is the firm’s constant proportion of assets paid in dividends per unit time. The value of the firm’s assets are assumed to follow the process

$$
\frac{dA}{A} = (\mu - \delta) dt + \sigma dz
$$

(18.1)

where $\mu$ denotes the instantaneous expected rate of return on the firm’s assets and $\sigma$ is the constant standard deviation of return on firm assets. Now let $D(t, T)$ be the date $t$ market value of the firm’s debt that is promised the payment of $B$ at date $T$. It is assumed that when the debt matures, the firm pays the promised amount to the debtholders if there is sufficient asset value to do so. If not, the firm defaults (bankruptcy occurs) and the debtholders take
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ownership of all of the firm’s assets. Hence, the payoff to debtholders at date $T$ can be written as

$$
D(T, T) = \min [B, A(T)] \\
= B - \max [0, B - A(T)]
$$

(18.2)

From the second line in equation (18.2), we see that the payoff to the debtholders equals the promised payment, $B$, less the payoff on a European put option written on the firm’s assets and having exercise price equal to $B$. Hence, if we make the usual “frictionless” market assumptions, then the current market value of the debt can be derived to equal the present value of the promised payment less the value of a put option on the dividend-paying assets.\(^2\) If we let $P(t, T)$ be the current date $t$ price of a default-free zero-coupon bond that pays $1$ at date $T$ and assume that the default-free term structure satisfies the Vasicek model as specified earlier in (9.28) to (9.30), then using Chapter 9’s results on the pricing of options when interest rates are random, we obtain:

$$
D(t, T) = P(t, T) B - P(t, T) BN(-h_2) + e^{-\delta \tau} AN(-h_1) \\
= P(t, T) BN(h_2) + e^{-\delta \tau} AN(-h_1)
$$

(18.3)

where $h_1 = [\ln \left( e^{-\delta \tau} A / (P(t, T) B) \right) + \frac{1}{2} \nu^2] / \nu$, $h_2 = h_1 - \nu$, and $\nu(\tau)$ is given in (9.48). Note that if the default-free term structure is assumed to be deterministic, then we have the usual Black-Scholes value for $\nu = \sigma \sqrt{\tau}$. The promised yield to maturity on the firm’s debt, denoted $R(t, T)$, can be calculated from (18.3) as $R(t, T) = \frac{1}{\tau} \ln [B/D(t, T)]$. Also, its credit spread can be computed as $R(t, T) - \frac{1}{\tau} \ln [1/P(t, T)]$.

\(^2\)One needs to assume that the risk of the firm’s assets, as determined by the $dz$ process, is a tradeable risk, so that a Black-Scholes hedge involving the firm’s debt can be constructed.
Based on this result, one can also solve for the market value of the firm’s shareholder’s equity, which we denote as \( E(t) \). In the absence of taxes and other transactions costs, the value of investors’ claims on the firm’s assets, \( D(t,T) + E(t) \) must equal the total value of the firm’s assets, \( A(t) \). This allows us to write

\[
E(t) = A(t) - D(t,T) \tag{18.4}
\]

\[
= A - P(t,T) BN(h_2) - e^{-\delta \tau} AN(-h_1)
\]

\[
= A \left[ 1 - e^{-\delta \tau} N(-h_1) \right] - P(t,T) BN(h_2)
\]

Shareholder’s equity is similar to a call option on the firm’s assets in the sense that at the debt’s maturity date, equity holders receive the payment \( \max[A(T) - B, 0] \). Shareholders’ limited liability gives them the option of receiving the firm’s residual value when it is positive. However, shareholders’ equity differs from the standard European call option if the firm pays dividends prior to the debt’s maturity. As is reflected in the first term in the last line of (18.4), the firm’s shareholders, unlike the holders of standard options, receive these dividends.

Chapter 12 of Robert Merton’s book (Merton 1992) gives an in-depth analysis of the comparative statics properties of the debt and equity formulas similar to equations (18.3) and (18.4), as well as the firm’s credit spread. Note that an equity formula such as (18.4) can be useful because for firms that have publicly-traded shareholders’ equity, observation of the firm’s market value of equity and its volatility can be used to infer the market value and volatility of the firm’s assets. The market value and volatility of the firm’s assets can then be used as inputs into (18.3) so that the firm’s default risky debt can then be valued. Such an exercise based on the Merton model has been done by the credit-rating
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firm Moody’s KMV to forecast corporate defaults.\(^3\)

The Merton model’s assumption that the firm has a single issue of zero-coupon debt is unrealistic, since it is commonly the case that firms have multiple coupon-paying debt issues with different maturities and different seniorities in the event of default. Modeling multiple debt issues and determining the point at which an asset deficiency triggers default is a complex task.\(^4\) In response, some research has taken a different tack by assuming that when the firm’s assets hits a lower boundary, default is triggered. This default boundary is presumed to bear a monotonic relation to the firm’s total outstanding debt. With the initial value of the firm’s assets exceeding this boundary, determining future default amounts to computing the first passage time of the assets through this boundary.

Francis Longstaff and Eduardo Schwartz (Longstaff and Schwartz 1995) develop such a model following the earlier work of Fischer Black and John Cox (Black and Cox 1976). They assume a default boundary that is constant over time and, when assets sink to the level of this boundary, bondholders are assumed to recover an exogenously given proportion of their bonds’ face values. This contrasts with the Merton model where, in the case of default, bondholders recover \(A(T)\), the stochastic value of firm assets at the bond’s maturity date, which results in a loss of \(B - A(T)\). In the Longstaff-Schwartz model, possible default occurs at a stochastic date, say \(\tau\), defined by the first (passage) time that \(A(\tau) = k\), where \(k\) is the pre-determined default boundary. Bondholders are assumed to recover \(\delta P(\tau, T)B\), where \(\delta < 1\) is the recovery rate equaling a proportion of the market value of an otherwise equivalent default-free bond,

\(^3\)For a description of the KMV application of the Merton model for forecasting defaults, see (Crosbie and Bohn 2002). Alan Marcus and Israel Shaked (Marcus and Shaked 1984) apply the Merton model to analyzing the default risk of commercial banks.

\(^4\)A study by Edward Jones, Scott Mason, and Eric Rosenfeld (Jones, Mason, and Rosenfeld 1984) is an example.
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This exogenous recovery rate, \( \delta \), is permitted to differ for bonds of different maturity and seniority characteristics and might be estimated from the historical recovery rates of different types of bonds.

Pierre Collin-Dufresne and Robert Goldstein (Collin-Dufresne and Goldstein 2001) modify the Longstaff-Schwartz model to permit a firm’s default boundary to be stochastic. Motivated by the tendency of firms to target their leverage ratios by partially adjusting their debt and equity over time, Collin-Dufresne and Goldstein permit the ratio of firm assets to firm debt (the default-boundary) to follow a mean reverting process with default triggered when this ratio declines to unity. Chunsheng Zhou (Zhou 2001) modifies the Longstaff Schwartz model in another direction by allowing the firm’s assets to follow a mixed jump-diffusion process. While these "first passage time" models seek to provide more realism than the more simple Merton model, they come at the cost of requiring numerical, rather than closed-form, solutions.

For firms with complicated debt structures, these first passage time models simplify the determination of default by assuming it occurs when a firm’s assets sink to a specified boundary. The interaction between default and the level and timing of particular promised bond payments are not directly modeled, except in so far as they affect the specification of the default boundary. In the next section, we consider the reduced form approach which goes a step further by not directly modeling either the firm’s assets or its overall debt level.

\[ P(\tau, T)B \] is the market value of a zero-coupon bond paying the face value of \( B \) at date \( T \). However, Longstaff and Schwartz do not limit their analysis to defaultable zero coupon bonds. Indeed, they value both fixed and floating coupon bonds assuming a Vasicek model of the term structure. Hence, in general, recovery equals a fixed proportion, \( \delta \), of the market value of an otherwise equivalent default-free (fixed or floating rate) bond.

More precisely, they assume that the risk-neutral process for the log of the ratio of firm debt to assets, say \( l(t) = \ln [k(t)/A(t)] \), follows an Ornstein-Uhlenbeck process. For an example of a model displaying mean-reverting leverage in the context of commercial bank defaults, see (Pennacchi 2005).

An exception is the closed-form solutions obtained by Stijn Claessens and George Pennacchi (Claessens and Pennacchi 1996) who model default-risky sovereign debt such as Brady bonds.
18.2 The Reduced-Form Approach

With the reduced form method, default need not be tied directly to the dynamics of a firm’s assets and liabilities. As a result, this approach may provide less insight regarding the economic fundamentals of default. However, because reduced form models generate default based on an exogenous Poisson process, they may better capture the effects on default of additional unobserved factors and provide richer dynamics for the term structure of credit spreads.\(^8\) Reduced form modeling also can be convenient because, as will be shown, defaultable bonds are valued using techniques similar to those used to value default-free bonds.

To illustrate reduced form modeling, we begin by analyzing a defaultable zero-coupon bond and, later, generalize the results to multiple-payment bonds. As in the previous section, let \(D(t,T)\) be the date \(t\) value of a default-risky zero-coupon bond that promises to pay \(B\) at its maturity date of \(T\). However, unlike previous section’s structural models where default is necessarily linked to the dynamics of the firm’s capital structure, we now assume that a possible default event depends on an exogenous process that may or may not depend directly on the firm’s capital structure. Default for a particular firm’s bond is modeled as a Poisson process with a time-varying default intensity. Conditional on default having not occurred prior to date \(t\), the instantaneous probability of

\(^8\)In most structural models, (Zhou 2001) is a notable exception, a firm’s assets are assumed to follow a diffusion process which has a continuous sample path. An implication of this is that default becomes highly unlikely for short horizons if the firm currently has a substantial difference between assets and liabilities. Hence, these models generate very small credit spreads for the short-maturity debt of credit-worthy corporation, counter to empirical evidence that finds more significant spreads. Small spreads occur because default over a short horizon cannot come as a sudden surprise. This is not the case with reduced form models where sudden default is always possible due its Poisson nature. Hence, these models can more easily match the significant credit spreads on short-term corporate debt. Darrell Duffie and David Lando (Duffie and Lando 2001) present a structural model where investors have less (accounting) information regarding the value of a firm’s assets than do the firm’s insiders. Hence, like the jump-diffusion model of Chunsheng Zhou (Zhou 2001), investors’ valuation of the firm’s assets can take discrete jumps when inside information is revealed. This model generates a Poisson default intensity equivalent to a particular reduced form model.
default during the interval \((t, t + dt)\) is denoted \(\lambda(t) dt\) where \(\lambda(t)\) is the physical default intensity or “hazard rate” and is assumed to be non-negative.\(^9\) Note from this definition, the physical probability of the firm not defaulting (that is, surviving) over the time interval from \(t\) to \(\tau\), where \(t < \tau \leq T\), is

\[
E_t \left[ e^{-\int_t^\tau \lambda(u) du} \right]
\]

(18.5)

18.2.1 A Zero-Recovery Bond

To determine \(D(t, T)\), an assumption must be made regarding the payoff received by bondholders should the bond default. We begin by assuming that bondholders recover nothing should the bond default and, later, we generalize this assumption to permit a possible non-zero recovery value. With zero recovery, the bondholders’ date \(T\) payoff is either \(D(T, T) = B\) if there is no default or \(D(T, T) = 0\) if default has occurred over the interval from \(t\) to \(T\). Applying risk-neutral pricing, the date \(t\) value of the zero-recovery bond, denoted \(D_Z(t, T)\), can be written as

\[
D_Z(t, T) = \hat{E}_t \left[ e^{-\int_t^T r(u) du} D(T, T) \right]
\]

(18.6)

where \(r(t)\) is the date \(t\) instantaneous default-free interest rate, and \(\hat{E}_t [\cdot]\) is the date \(t\) risk-neutral expectations operator. To compute this expression, we need to determine the expression for \(D(T, T)\) in terms of the risk-neutral default intensity, rather than the physical default intensity. The risk-neutral default intensity will account for the market price risk associated with the Poisson

\(^9\)Recall that in Chapter 11 we modeled jumps in asset prices as following a Poisson process with jump intensity \(\lambda\). Here, a one-time default follows a Poisson process, and its intensity is explicitly time-varying.
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arrival of a default event.

To understand the role of default risk, suppose that both the default-free term structure and the firm’s default intensity depend on a set of \( n \) state variables, \( x_i, \ i = 1, \ldots, n \), that follow the multi-variate Markov diffusion process\(^{10}\)

\[
dx = a(t, x) \, dt + b(t, x) \, dz
\]

(18.7)

where \( x = (x_1 \ldots x_n)' \), \( a(t, x) \) is a \( nx1 \) vector, \( b(t, x) \) is a \( nxn \) matrix, and \( dz = (dz_1 \ldots dz_n)' \) is an \( n \times 1 \) vector of independent Brownian motion processes so that \( dz_i \, dz_j = 0 \) for \( i \neq j \). As in the previous chapter, \( x(t) \) includes macro-economic factors that affect the default-free term structure, but it now also includes firm-specific factors that affect the likelihood of default for the particular firm. Similar to (17.8), the stochastic discount factor for pricing the firm’s default risky bond will be of the form

\[
dM/M = -r(t, x) \, dt - \Theta(t, x)' \, dz - \gamma(t, x) \, dq - \lambda(t, x) \, dt
\]

(18.8)

where \( \Theta(t, x) \) is an \( nx1 \) vector of the market prices of risk associated with the elements of \( dz \) and \( \gamma(t, x) \) is the market price of risk associated with the actual default event which occurs when the Poisson process \( q(t) \) jumps from 0 (the no default state) to 1 (the absorbing default state) at which time \( dq = 1 \).\(^{11}\) The risk-neutral default intensity, \( \hat{\lambda}(t, x) \), is then given by

\[
\hat{\lambda}(t, x) = [1 - \gamma(t, x)] \lambda(t, x)
\]

Note that in this modeling context, default is "doubly-

\(^{10}\)For concreteness our presentation assumes an equilibrium Markov state variable environment. However, much of our results on reduced form pricing of defaultable bonds carry over to a non-Markov, no-arbitrage context, such as the Heath-Jarrow-Morton framework. See (Duffie and Singleton 1999).

\(^{11}\)Recall from the discussion in Chapter 11 that jumps in an asset’s value, as would occur when a bond defaults, cannot always be hedged. Thus, in general it may not be possible to determine \( \gamma(t, x) \) based on a no arbitrage restriction. This market price of default risk may need to be determined from an equilibrium model of investor preferences.
stochastic." Default depends on the Brownian motion vector $\mathbf{dz}$ that drives $\mathbf{x}$ and determines how the likelihood of default, $\widehat{\lambda}(t, x)$, changes over time, but it also depends on the Poisson process $dq$ which determines the arrival of default. Hence, default risk reflects two types of risk-premia, $\Theta(t, x)$ and $\gamma(t, x)$.

Based on the calculation of survival probability in (18.5), the value of the zero-recovery defaultable bond is

$$D_Z(t, T) = b E_t \left[ e^{-R_T} \cdot e^{\int_t^T \left[ r(u) + b \lambda(u) \right] du} \right] B (18.9)$$

Equation (18.9) shows that valuing this zero-recovery defaultable bond is similar to valuing a default free bond except that we use the discount rate of $r(u) + \widehat{\lambda}(u)$ rather than just $r(u)$. Given specific functional forms for $r(t, x)$, $\lambda(t, x)$, and the risk-neutral state variable process (specifications of (18.7) and $\Theta(t, x)$), the expression in (18.9) can be computed.

### 18.2.2 Specifying Recovery Values

The value of a bond that has a possibly non-negative recovery value in the event of default equals the value in (18.9) plus the present value of the amount recovered in default. Suppose that if the bond defaults at date $\tau$ where $t < \tau \leq T$, bondholders recover an amount $w(\tau, x)$ at date $\tau$. Now note the the risk-neutral probability density of defaulting at time $\tau$ is

$$e^{-\int_t^\tau \widehat{\lambda}(u) du} \widehat{\lambda}(\tau)$$

In (18.10), $\widehat{\lambda}(\tau)$ is discounted by $\exp \left[ - \int_t^\tau \widehat{\lambda}(u) du \right]$ because default at date $\tau$ is conditioned on not having defaulted previously. Therefore, the present value of recovery in the event of default, $D_R(t, T)$ is computed by integrating the expected discounted value of recovery over all possible default dates from $t$ to
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\begin{equation}
T:

D_R(t,T) = \hat{E}_t \left[ \int_t^T e^{-\int_t^\tau r(u)du} w(\tau) e^{-\int_t^\tau \hat{\lambda}(u)du} \hat{\lambda}(\tau) d\tau \right]
= \hat{E}_t \left[ \int_t^T e^{-\int_t^\tau [r(u)+\hat{\lambda}(u)]du} \hat{\lambda}(\tau) w(\tau) d\tau \right]
\end{equation}

Putting this together with (18.9) gives the bond’s total value, \( D(t,T) = D_Z(t,T) + D_R(t,T) \), as

\begin{equation}
D(t,T) = \hat{E}_t \left[ e^{-\int_t^T [r(s)+\hat{\lambda}(s)]ds} B + \int_t^T e^{-\int_t^\tau [r(s)+\hat{\lambda}(s)]ds + \hat{\lambda}(\tau) w(\tau) d\tau} \right]
\end{equation}

Recovery Proportional to Par Value

Let us consider some particular specifications for \( w(\tau, x) \). One assumption used by several researchers is that bondholders recover at the default date \( \tau \) a proportion of the bond’s face or par value, that is, \( w(\tau, x) = \delta(\tau, x) B \) where \( \delta(\tau, x) \) is usually assumed to be a constant, say \( \bar{\delta} \). In this case (18.11) can be written as

\begin{equation}
D_R(t,T) = \bar{\delta} B \int_t^T k(t,\tau) d\tau
\end{equation}

where

\begin{equation}
k(t,\tau) \equiv \hat{E}_t \left[ e^{-\int_t^\tau [r(u)+\hat{\lambda}(u)]du} \hat{\lambda}(\tau) \right]
\end{equation}

has a closed-form solution when \( r(u, x) \) and \( \hat{\lambda}(u, x) \) are affine functions of \( x \) and the vector \( x \) in (18.7) has a risk-neutral process that is also affine. In this case, the recovery value in (18.13) can be computed by numerical integration of

\textsuperscript{12} Work by Darrell Duffie (Duffie 1998), David Lando (Langer 1998), and Dilip Madan and Haluk Unal (Madan and Unal 1998) make this assumption. As reported by Gregory Duffie (Duffie 1999), the recovery rate, \( \bar{\delta} \), estimated by Moody’s for senior unsecured bondholders is approximately 44 percent.

\textsuperscript{13} This is shown in (Duffie, Pan, and Singleton 2000).
Recovery Proportional to Par Value, Payable at Maturity

An alternative recovery assumption is that if default occurs at date \( \tau \), the bondholders recover a proportion \( \delta(\tau, x) \) of the bond’s face value, \( B \), payable at the maturity date \( T \). This is equivalent to assuming that the bondholders recover a proportion \( \delta(\tau, x) \) of the market value of a default-free discount bond paying \( B \) at date \( T \), that is, \( w(\tau, x) = \delta(\tau, x) P(\tau, T) B \). Under this assumption (18.11) becomes

\[
D_R(t, T) = \mathbb{E}_t \left[ \int_t^T e^{-\int_t^u \lambda(u) du} \delta(\tau, x) e^{-\int_t^\tau r(u) du} B d\tau \right] - \mathbb{E}_t \left[ \int_t^T e^{-\int_t^u \lambda(u) du} \left( 1 - e^{-\int_t^\tau r(u) du} \right) \delta B \right] = \delta BP(t, T) - \delta D_Z(t, T) \quad (18.16)
\]

For the specific case of \( \delta(\tau, x) = \delta \), a constant, this expression can be simplified by noting that the term \( \int_t^T \exp \left[ -\int_t^\tau \lambda(u) du \right] \delta(\tau, x) d\tau \) is the total risk-neutral probability of default for the period from date \( t \) to the maturity date \( T \). Therefore it must equal \( 1 - \exp \left[ -\int_t^T \lambda(u) du \right] \), that is, one minus the probability of surviving over the same period. Making this substitution and using (18.9) we have

\[
D_R(t, T) = \mathbb{E}_t \left[ e^{-\int_t^\tau r(u) du} \left( 1 - e^{-\int_t^\tau \lambda(u) du} \right) \delta B \right] = \mathbb{E}_t \left[ e^{-\int_t^\tau r(u) du} - e^{-\int_t^\tau [r(u) + \lambda(u)] du} \right] \delta B = \delta BP(t, T) - \delta D_Z(t, T) \quad (18.16)
\]

\[\text{14} \text{ This specification has been studied by Robert Jarrow and Stuart Turnbull (Jarrow and Turnbull 1995) and David Lando (Langer 1998).}\]
Therefore, the total value of the bond is

\[
D (t, T) = D_Z (t, T) + D_R (t, T) = (1 - \delta) D_Z (t, T) + \delta BP (t, T) \tag{18.17}
\]

Hence, this recovery assumption amounts to requiring only a solution for the value of a zero-recovery bond.

**Recovery Proportional to Market Value**

Let us consider one additional recovery assumption analyzed by Darrell Duffie and Kenneth Singleton (Duffie and Singleton 1999). When default occurs, bondholders are assumed to recover a proportion of what was the bond’s market value just prior to default. This is equivalent to assuming that the bond’s market value jumps downward at the default date \( \tau \), suffering a proportional loss of \( L (\tau, x) \). Specifically, at default \( D (\tau^-, T) \) jumps to

\[
D (\tau^+, T) = w (\tau, x) = D (\tau^-, T) [1 - L (\tau, x)] \tag{18.18}
\]

By specifying a proportional loss in value at the time of default, the bond’s dynamics become similar to the jump-diffusion model of asset prices presented in Chapter 11. Treating the defaultable bond as a contingent claim and applying Itô’s lemma, its process prior to default is similar to equation (11.7):

\[
dD (t, T) / D (t, T) = (\alpha_D - \lambda k_D) dt + \sigma_D' dz + dq_D \tag{18.19}
\]

where \( \alpha_D \) and the \( nx1 \) vector \( \sigma_D \) are given by the usual Itô’s lemma expressions similar to (11.8) and (11.9). From (11.6) and (18.18), we have that when a jump occurs \( dq_D = [D (\tau^+, T) - D (\tau^-, T)] / D (\tau^-, T) = -L (\tau, x) \), so that we can write \( dq_D = -L (\tau, x) dq \). Also, from (11.10), \( k_D \), the expected jump size, is given by \( k_D (\tau^-) \equiv E_- [D (\tau^+, T) - D (\tau^-, T)] / D (\tau^-, T) = -L (\tau, x) \), so
that the drift term in (18.19) becomes $\alpha_D + \lambda(t, x) L(t, x)$.

Now since under the risk-neutral measure, the defaultable bond’s total expected rate of return, $\alpha_D$, equals the instantaneous maturity default-free rate, $r(t)$, we can write the bond’s risk-neutral process prior to default as

$$dD(t, T)/D(t, T) = \left(r(t, x) + \hat{\lambda}(t, x) \tilde{L}(t, x)\right) dt + \sigma'_D d\tilde{z} - \tilde{L}(t, x) dq$$  (18.20)

where $\tilde{L}(t, x)$ is the risk-neutral expected proportional loss given default.\footnote{As with the risk-neutral default intensity, $\hat{\lambda}(t, x)$, there may be a market price of recovery risk associated with $\tilde{L}(t, x)$ that distinguishes it from the physical expected loss at default, $L(t, x)$. This market price of recovery risk cannot, in general, be determined from a no arbitrage restriction because recovery risk may be unhedgeable. Most commonly, modelers simply posit functional forms for risk-neutral variables in order to derive formulas for defaultable bond values. Differences between risk-neutral default intensities and losses at default and their physical counterparts might be inferred based on the market prices of defaultable bonds and historical (physical) default and recovery rates.} The intuition of (18.20) is that because the bond has a risk-neutral expected loss given default of $\hat{\lambda}(t, x) \tilde{L}(t, x)$, and the risk-neutral instantaneous probability of default ($dq = 1$) is $\hat{\lambda}(t, x)$, when the bond does not default it must earn an excess expected return of $\hat{\lambda}(t, x) \tilde{L}(t, x)$ to make its unconditional risk-neutral expected return equal $r(t)$. Based on a derivation similar to that used to obtain (11.16) and (17.6), one can show that the defaultable bond’s value satisfies the equilibrium partial differential equation

$$\frac{1}{2} \text{Trace} \left[ b(t, x) b(t, x)' D_{xx} \right] + \tilde{a}(t, x)' D_x - R(t, x) D + D_t = 0$$  (18.21)

where $D_x$ denotes the $nx1$ vector of first derivatives of $D(t, x)$ with respect to each of the factors and, similarly, $D_{xx}$ is the $nxn$ matrix of second order mixed partial derivatives. In addition, $\tilde{a}(t, x) = a(t, x) - b(t, x) \Theta$ is the risk-neutral drift of the factor process (18.7) and $R(t, x) \equiv r(t, x) + \hat{\lambda}(t, x) \tilde{L}(t, x)$ is the defaultable bond’s risk-neutral drift in the process (18.20). Now note that should
the bond reach the maturity date, $T$, without defaulting, then $D(T, T) = B$ which determines the boundary condition for (18.21). The PDE (18.21) is in the form of a PDE for a standard contingent claim except that $R(t, x)$ has replaced $r(t, x)$ in the standard PDE. This insight allows us to write the PDE's Feynman-Kac solution as:

$$D(t, T) = \hat{E}_t \left[ e^{-\int_t^T R(u, x) du} \right] B$$

(18.22)

where $R(t, x) \equiv r(t, x) + \hat{\lambda}(t, x) \hat{L}(t, x)$ can be viewed as the "default-adjusted" discount rate. The product $s(t, x) \equiv \hat{\lambda}(t, x) \hat{L}(t, x)$ has the interpretation as the "credit spread" on an instantaneous-maturity defaultable bond. Since $\hat{\lambda}(t, x)$ and $\hat{L}(t, x)$ are not individually identified in (18.22), when implementing this formula we can simply specify a single functional form for $s(t, x)$.

18.2.3 Examples

Because default intensities and/or credit spreads must be non-negative, a popular stochastic process for modeling these variables is the mean-reverting, square-root process used in the term structure model of John Cox, Jonathan Ingersoll, and Stephen Ross (Cox, Ingersoll, and Ross 1985b). To take a very simple example, suppose that $x = (x_1, x_2)'$ is a two-dimensional vector, $\hat{a}(t, x) = (\kappa_1 (x_1 - \mu_1) \quad \kappa_2 (x_2 - \mu_2))'$, and $b(t, x)$ is a diagonal matrix with first and second diagonal elements of $\sigma_1 \sqrt{x_1}$ and $\sigma_2 \sqrt{x_2}$, respectively. If one assumes $r(t, x) = x_1(t)$ and $\hat{\lambda}(t, x) = x_2(t)$, this has the implication that the default-free term structure and the risk-neutral default intensity are independent. Arguably, this is unrealistic since empirical work has found a negative correlation

\footnote{Recall from Chapter 10 that (10.14) was shown to be the Feynman-Kac solution to the Black-Scholes PDE (10.7). See Durrett Duffie and Kenneth Singleton (Duffie and Singleton 1999) for an alternative derivation of (18.22) that does not involve specification of factors or the bond’s PDE.}
between default-free interest rates and the likelihood of corporate defaults.\footnote{This evidence is presented in work by Gregory Duffee ((Duffee 1999)) and Pierre Collin-Dufresne and Bruno Solnik ((Collin-Dufresne and Solnik 2001)).}

Allowing for non-zero correlation between \( r(t, x) \) and \( \tilde{\lambda}(t, x) \) while restricting each to be positive is certainly feasible but comes at the cost of requiring numerical, rather than closed-form, solutions for defaultable bond values.\footnote{For models with more flexible correlation structures that require numerical solutions, see examples given by Darrell Duffie and Kenneth Singleton (Duffie and Singleton 1999). Some research has dropped the restriction that \( r(t) \) and \( \tilde{\lambda}(t) \) (or \( s(t) = \tilde{\lambda}(t) L(t) \)) be positive by assuming these variables follow multi-variate affine Gaussian processes. This permits general correlation between default-free interest rates and default intensities as well as closed-form solutions for defaultable bonds. The model in work by C.V.N. Krishnan, Peter Ritchken, and James Thomson ((Krishnan, Ritchken, and Thomson 2004)) is an example of this.}

Hence, for simplicity of presentation, we maintain the independence assumption.

With \( r(t, x) = x_1(t) \) and denoting \( \pi_1 = \pi \), we obtain the Cox, Ingersoll, and Ross formula for the value of a default-free discount bond:\footnote{The formula in (18.23) to (18.25) is the same as (13.51) to (13.53) except that it is written in terms of the parameters of the risk-neutral, rather than physical, process for \( r(t) \). Hence, relative to our earlier notation, \( \kappa_1 = \kappa + \psi \) where the market price of interest rate risk equals \( \theta(t) = -\psi \sqrt{r} / \sigma_1 \).}

\[
P(t, T) = A_1(\tau) e^{-B_1(\tau)r(t)} \tag{18.23}
\]

where

\[
A_1(\tau) = \left[ \frac{2\theta_1 e^{(\theta_1 + \kappa_1)\tau}}{(\theta_1 + \kappa_1)(e^{\theta_1\tau} - 1) + 2\theta_1} \right]^{2\kappa_1 \tau / \sigma_1^2}, \tag{18.24}
\]

\[
B_1(\tau) = \frac{2(e^{\theta_1\tau} - 1)}{(\theta_1 + \kappa_1)(e^{\theta_1\tau} - 1) + 2\theta_1}, \tag{18.25}
\]

and \( \theta_1 = \sqrt{\kappa_1^2 + 2\sigma_1^2} \). Also with \( \tilde{\lambda}(t, x) = x_2(t) \) and denoting \( \pi_2 = \pi \), then based on (18.9) and the assumed independence of \( r(t) \) and \( \tilde{\lambda}(t) \) we can write...
the value of the zero-recovery bond as

$$D_Z(t,T) = \hat{E}_t \left[ e^{-\int_t^T r(s) + \hat{\lambda}(s) ds} \right] B$$

$$= \hat{E}_t \left[ e^{-\int_t^T r(s) ds} \right] \hat{E}_t \left[ e^{-\int_t^T \hat{\lambda}(s) ds} \right] B$$

$$= P(t,T) V(t,T) B$$  \hspace{1cm} (18.26)

where

$$V(t,T) = A_2(\tau) e^{-B_2(\tau)\hat{\lambda}(t)}$$  \hspace{1cm} (18.27)

and where $A_2(\tau)$ is the same as $A_1(\tau)$ in (18.24) and $B_2(\tau)$ is the same as $B_1(\tau)$ in (18.25) except that $\kappa_2$ replaces $\kappa_1$, $\sigma_2$ replaces $\sigma_1$, $\bar{\kappa}$ replaces $\kappa$, and $\theta_2 \equiv \sqrt{\kappa_2^2 + 2\sigma_2^2}$ replaces $\theta_1$.

If we assume that recovery is a fixed proportion, $\delta$, of par value, payable at maturity, then based on (18.17) the value of the defaultable bond equals

$$D(t,T) = (1 - \delta) D_Z(t,T) + \delta B P(t,T)$$

$$= \left[ \delta + (1 - \delta) V(t,T) \right] P(t,T) B$$  \hspace{1cm} (18.28)

$V(t,T)$ in (18.27) is analogous to a bond price in the standard Cox, Ingersoll, and Ross term structure model, and as such it will be inversely related to $\hat{\lambda}(t)$ and strictly less than 1 whenever $\hat{\lambda}(t)$ is strictly positive, which can be ensured when $2\kappa_2\bar{\kappa} \geq \sigma_2^2$. Thus, (18.28) confirms that the defaultable bond’s value declines as its risk-neutral default intensity rises.

A slightly different defaultable bond formula can be obtained when recovery is assumed to be proportional to market value and $s(t,x) \equiv \hat{\lambda}(t,x) \hat{L}(t,x) = x_2$
with the notation $\tau_2 = \tau$. In this case, (18.22) becomes

$$D(t,T) = \tilde{E}_t \left[ e^{-\int_t^T r(u) + s(u) du} \right] B$$

$$= \tilde{E}_t \left[ e^{-\int_t^T r(u) du} \right] \tilde{E}_t \left[ e^{-\int_t^T s(u) du} \right] B$$

$$= P(t,T) S(t,T) B$$

(18.29)

where

$$S(t,T) = A_2(\tau) e^{-B_2(\tau)s(t)}$$

(18.30)

and where $A_2(\tau)$ is the same as $A_1(\tau)$ in (18.24) and $B_2(\tau)$ is the same as $B_1(\tau)$ in (18.25) except that $\kappa_2$ replaces $\kappa_1$, $\sigma_2$ replaces $\sigma_1$, $\tau$ replaces $\tau$, and $\theta_2 \equiv \sqrt{\kappa_2^2 + 2\sigma_2^2}$ replaces $\theta_1$. This defaultable bond is priced similar to a default-free bond except that the instantaneous maturity interest rate, $R(t) = r(t) + s(t)$ is now the sum of two non-negative square root processes. Hence the defaultable bond is inversely related to $s(t)$ and can be strictly less than the default-free bond as $s(t)$ can always be positive when $2\kappa_2 \tau \geq \sigma_2^2$.

Valuing the defaultable coupon bond of a particular issuer (e.g., corporation) is straightforward given the preceding analysis of defaultable zero-coupon bonds. Suppose that the issuer’s coupon bond promises $n$ cashflows, with the $i^{th}$ promised cashflow being equal to $c_i$ and being paid at date $T_i > t$. Then the value of this coupon bond in terms of our zero-coupon bond formulas is

$$\sum_{i=1}^n D(t,T_i) \frac{c_i}{B}$$

(18.31)

Our results can also be applied to valuing credit derivatives. A credit default swap is a popular credit derivative that typically has the following structure. One party, the protection buyer, makes periodic payments until the contract’s maturity date as long as a particular issuer, bond, or loan does not default.
The other party, the protection seller, receives these payments in return for paying the difference between the bond or loan’s par value and its recovery value if default occurs prior to the maturity of the swap contract. At the initial agreement date of this swap contract, the periodic payments are set such that the initial contract has a zero market value.

We can use our previous analysis to value each side of this swap. Let the contract specify equal period payments of $c$ at future dates $t + \Delta, t + 2\Delta, \ldots, t + n\Delta$.\footnote{A period of $\Delta = \text{one-half year}$ is common since these payments often coincide with an underlying coupon bond making semi-annual payments.} Then recognizing that these payments are contingent on default not occurring and that they have zero value following a possible default event, their market value equals

$$
\frac{c}{B} \sum_{i=1}^{n} D_Z (t, t + i\Delta)
$$

(18.32)

where $D_Z (t, T)$ is the value of the zero-recovery bond given in (18.9). If we let $w (\tau, x)$ be the recovery value of the defaultable bond (or loan) underlying the swap contract, then assuming this bond’s maturity date is $T \geq t + n\Delta$, the value of the swap protection can be computed similar to (18.11) as

$$
\tilde{E}_t \left[ \int_{t}^{t + n\Delta} e^{-\int_{t}^{\tau} [r(u) + \tilde{\lambda}(u)] du} \tilde{\lambda}(\tau) [B - w (\tau)] d\tau \right]
$$

(18.33)

The protection seller’s payment in the event of default, $B - w (\tau)$, is often simplified by assuming recovery is a fixed proportion of par value, that is, $B - w (\tau) = B - \delta B = B (1 - \delta)$. For this special case, (18.33) becomes

$$
B (1 - \delta) \int_{t}^{t + n\Delta} k (t, \tau) d\tau
$$

(18.34)

where $k (t, \tau)$ is defined in (18.14). For given assumptions regarding the function forms of $r (t, x), \tilde{\lambda}(t, x)$, and $w (t, x)$, and the state variables $x$, the value
of the swap payments, $c$, can be determined that equates (18.32) to (18.33).

A general issue that arises when implementing the reduced-form approach to valuing risky debt is determining the proper current values $\hat{\lambda}(t)$, $s(t)$, or $w(t)$ that may not be directly observable. One or more of these default variables might be inferred by setting the actual market prices of one or more of an issuer’s bonds to their theoretical formulas. Then, based on the “implied” values of $\hat{\lambda}(t)$, $s(t)$, or $w(t)$, one can determine whether a given bond of the same issuer is over- or under-priced relative to other bonds. Alternatively, these implied default variables could be used to set the price of a new bond of the same issuer or a credit derivative (such as a default swap) written on the issuer’s bonds.

18.3 Summary

Research on credit risk has grown significantly in recent years, generating and generated by greater interest in credit risk management and credit derivatives. This chapter introduced the two main branches of modeling defaultable bond values. The structural approach derives default based on the interaction between a firm’s assets and liabilities. Potentially, it can improve our understanding between capital structure and corporate bond prices. In contrast, the reduced form method abstracts from specific aspects of a firm’s financial structure, but it can permit a more flexible modeling of default probabilities and may provide a better fit to the prices of an issuer’s bonds.

While, due to space constraints, this chapter has been limited to models of corporate defaults, the credit risk literature also encompasses additional topics such as consumer credit risk and the credit risk of (securitized) portfolios of loans and bonds. Interest by both academics and practitioners in the broad field of credit risk will undoubtedly continue.
18.4 Exercises

1. Consider the example given in the “structural approach” to modeling default risk. Maintain the assumptions made in the chapter but now suppose that a third party guarantees the firm’s debtholders that if the firm defaults, the debtholders will receive their promised payment of \( B \). In other words, this third-party guarantor will make a payment to the debtholders equal to the difference between the promised payment and the firm’s assets if default occurs. (Banks often provide such a guarantee in the form of a letter of credit. Insurance companies often provide such a guarantee in the form of bond insurance.)

What would be the fair value of this bond insurance at the initial date, \( t \)? In other words, what would be the competitive bond insurance premium to be charged at date \( t \)?

2. Consider a Merton (1974)-type “structural” model of credit risk. A firm is assumed to have shareholders’ equity and two zero-coupon bonds that both mature at date \( T \). The first bond is “senior” debt and promises to pay \( B_1 \) at maturity date \( T \) while the second bond is “junior” (or subordinated) debt and promises to pay \( B_2 \) at maturity date \( T \). Let \( A(t) \), \( D_1(t) \), and \( D_2(t) \) be the date \( t \) values of the firm’s assets, senior debt, and junior debt, respectively. Then the maturity value of the bonds are

\[
D_1(T) = \begin{cases} 
B_1 & \text{if } A(T) \geq B_1 \\
A(T) & \text{otherwise}
\end{cases}
\]
The firm is assumed to pay no dividends to its shareholders, and the value of shareholders’ equity at date \( T \), \( E (T) \), is assumed to be

\[
E (T) = \begin{cases} 
A (T) - (B_1 + B_2) & \text{if } A (T) \geq B_1 + B_2 \\
0 & \text{otherwise}
\end{cases}
\]

Assume that the value of the firm’s assets follows the process

\[
dA/A = \mu dt + \sigma dz
\]

where \( \mu \) denotes the instantaneous expected rate of return on the firm’s assets and \( \sigma \) is the constant standard deviation of return on firm assets. In addition, the continuously-compounded risk-free interest rate is assumed to be the constant \( r \). Let the current date be \( t \), and define the time until the debt matures as \( \tau \equiv T - t \).

2.a Give a formula for the current, date \( t \), value of shareholders equity, \( E (t) \).
2.b Give a formula for the current, date \( t \), value of the senior debt, \( D_1 (t) \).
2.c Using the results from 2.a and 2.b, give a formula for the current, date \( t \), value of the junior debt, \( D_2 (t) \).

3. Consider a portfolio of \( m \) different defaultable bonds (or loans), where the \( i^{th} \) bond has a default intensity of \( \lambda_i (t, x) \) where \( x \) is a vector of state variables that follows the multi-variate diffusion process in (18.7). Assume that the only source of correlation between the bonds’ defaults is through
their default intensities. Suppose that the maturity dates for the bonds all exceed date $T > t$. Write down the expression for the probability that none of the bonds in the portfolio defaults over the period from date $t$ to date $T$.

4. Consider the standard (plain vanilla) swap contract described in the previous chapter. In equation (17.67) it was shown that under the assumption that each party’s payments were default free, the equilibrium swap rate agreed to at the initiation of the contract, date $T_0$, equals

$$
s_{0,n} (T_0) = \frac{1 - P (T_0, T_{n+1})}{\tau \sum_{j=1}^{n+1} P (T_0, T_j)}
$$

where for this contract fixed interest rate coupon payments are exchanged for floating interest rate coupon payments at the dates $T_1, T_2, ..., T_{n+1}$ with $T_{j+1} = T_j + \tau$ and $\tau$ is the maturity of the LIBOR of the floating rate coupon payments. This swap rate formula is valid when neither of the parties have credit risk. Suppose, instead, that they both have the same credit risk, and it is equivalent to the credit risk reflected in LIBOR interest rates.\(^{21}\) Moreover, assume a reduced form model of default with recovery proportional to market value, so that the value of a LIBOR discount bond promising $\$1$ at maturity date $T_j$ is given by $(18.22)$:

$$
D (T_0, T_j) = \widehat{E}_{T_0}\left[ e^{-\int_{T_0}^{T_j} R(u, x) du} \right]
$$

where the default-adjusted instantaneous discount rate $R (t, x) \equiv r (t, x) + \lambda (t, x) \tilde{L} (t, x)$ is assumed to be the same for both parties. Assume that if default occurs at some date $\tau < T_{n+1}$, the counterparty whose position is "in-the-money" (whose position has positive value) suffers a proportional

\(^{21}\) Recall that LIBOR reflects the level of default risk for a large international bank.
loss of $L(\tau, \mathbf{x})$ in that position. Show that under these assumptions, the equilibrium swap rate is

$$s_{0,n}(T_0) = \frac{1 - D(T_0, T_{n+1})}{\tau \sum_{j=1}^{n+1} D(T_0, T_j)}.$$