Chapter 16

Asset Pricing with
Differential Information

The asset pricing models studied thus far have assumed that individuals have common information. In this chapter, we consider the situation where individuals can have different private information about an asset’s future payoff or value. Of particular interest is how a risky asset’s equilibrium price will be affected by this private information. We start by analyzing a model developed by Sanford Grossman (Grossman 1976). It shows how individuals’ information affects their demands for an asset, and, via these demands, how private information is contained in the asset’s equilibrium price. The model examines two equilibria: a “competitive,” but not fully rational, equilibrium; and a fully-revealing rational expectations equilibrium.

Following this, we examine an extension of the Grossman model that includes an additional source of uncertainty, namely, shifts in the supply of the risky asset. A model of this type was developed in a number of studies (Grossman and Stiglitz 1980), (Hellwig 1980), (Diamond and Verrecchia 1981), and (Grundy 1981).
and McNichols 1989). Importantly, in a rational expectations equilibrium this additional supply uncertainty makes the equilibrium asset price only partially reveal the private information of individuals.

We cover one additional model of a risky asset market that also possesses an equilibrium where private information is partially revealed. It is the seminal “market micro-structure” model developed by Albert "Pete" Kyle (Kyle 1985). This model assumes a market for a particular security in which one agent, the so-called "insider," has private information and trades with lesser-informed agents composed of a market maker and "noise" traders. The model solves for the strategic trading behavior of the insider and market maker and provides a theoretical framework for determining bid-ask spreads and the “market impact” of trades.

16.1 Equilibrium with Private Information

The model by Sanford Grossman (Grossman 1976) that we consider in this section examines how an investor’s private information about a risky asset’s future payoff affects her demand for that asset and, in turn, the asset’s equilibrium price. In addition, it takes account of the idea that a rational individual can learn about others’ private information from the risky asset’s price, a concept known as "price discovery." The model is based on the following assumptions.

16.1.1 Grossman Model Assumptions

A.1 Assets

This is a single-period portfolio choice problem. At the beginning of the period, traders can choose between a risk-free asset, which pays a known end-
of-period return (one plus the interest rate) of $R_f$, or a risky asset that has a beginning-of-period price of $P_0$ per share and has an end-of-period random payoff (price) of $P_1$ per share. The unconditional distribution of $P_1$ is assumed to be normally distributed as $N(m, \sigma^2)$. The aggregate supply of shares of the risky asset is fixed at $\bar{X}$, but the risk-free asset is in perfectly elastic supply.

A.2 Trader Wealth and Preferences

There are $n$ different traders. The $i^{th}$ trader has beginning-of-period wealth $W_{0i}$ and is assumed to maximize expected utility over end-of-period wealth, $\tilde{W}_{1i}$. Each trader is assumed to have constant absolute risk aversion (CARA) utility, but traders’ levels of risk aversion are permitted to differ. Specifically, the form of the $i^{th}$ trader’s utility function is assumed to be

$$U_i(\tilde{W}_{1i}) = -e^{-a_i \tilde{W}_{1i}}, \quad a_i > 0.$$  \hspace{1cm} (16.1)

A.3 Trader Information

At the beginning of the period, the $i^{th}$ trader observes $y_i$, which is a realized value from the noisy signal of the risky asset end-of-period value

$$\tilde{y}_i = \tilde{P}_1 + \tilde{\epsilon}_i$$  \hspace{1cm} (16.2)

where $\tilde{\epsilon}_i \sim N(0, \sigma_i^2)$ and is independent of $\tilde{P}_1$.

16.1.2 Individuals’ Asset Demands

Let $X_{fi}$ be the amount invested in the risk-free asset and $X_i$ be the number of shares of the risky asset chosen by the $i^{th}$ trader at the beginning of the period. Thus,

$$W_{0i} = X_{fi} + P_0 X_i.$$  \hspace{1cm} (16.3)
The \(i\)th trader’s wealth accumulation equation can be written as
\[
\tilde{W}_{1i} = R_f W_{0i} + \left( \tilde{P}_1 - R_f P_0 \right) X_i. (16.4)
\]
Denote \(I_i\) as the information available to the \(i\)th trader at the beginning of the period. The trader’s maximization problem is then
\[
\max_{X_i} E \left[ U_i(\tilde{W}_{1i}) \mid I_i \right] = \max_{X_i} E \left[ -e^{-a_i (R_f W_{0i} + [\tilde{P}_i - R_f P_0] X_i)} \mid I_i \right]. (16.5)
\]
Since \(\tilde{W}_{1i}\) depends on \(\tilde{P}_i\), it is normally distributed, and, due to the exponential form of the utility function, (16.5) is the moment generating function of a normal random variable. Therefore, as we have seen earlier in the context of mean-variance analysis, the maximization problem is equivalent to
\[
\max_{X_i} \left\{ E \left[ \tilde{W}_{1i} \mid I_i \right] - \frac{1}{2} a_i \text{Var} \left[ \tilde{W}_{1i} \mid I_i \right] \right\} (16.6)
\]
or
\[
\max_{X_i} \left\{ X_i \left( E \left[ \tilde{P}_i \mid I_i \right] - R_f P_0 \right) - \frac{1}{2} a_i X_i^2 \text{Var} \left[ \tilde{P}_i \mid I_i \right] \right\}. (16.7)
\]
The first-order condition with respect to \(X_i\) then gives us the optimal number of shares held in the risky asset:
\[
X_i = \frac{E \left[ \tilde{P}_i \mid I_i \right] - R_f P_0}{a_i \text{Var} \left[ \tilde{P}_i \mid I_i \right]} . (16.8)
\]
Note that the CARA utility assumption results in the investor’s demand for the risky asset being independent of wealth. This simplifies the derivation of the risky asset’s equilibrium price.
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16.1.3 A Competitive Equilibrium

Now consider an equilibrium in which each trader uses his knowledge of the unconditional distribution of \( \tilde{P}_1 \) along with the conditioning information from his private signal, \( y_i \), so that \( I_i = \{ y_i \} \). Then using Bayes rule and the fact that \( \tilde{P}_1 \) and \( \tilde{y}_i \) are jointly normally distributed with a squared correlation \( \rho_i \equiv \frac{\sigma^2}{\sigma^2 + \sigma^2_i} \), the \( i \)th trader’s conditional expected value and variance of \( \tilde{P}_1 \) are

\[
E \left[ \tilde{P}_1 \mid I_i \right] = m + \rho_i (y_i - m) \tag{16.9}
\]

\[
\text{Var} \left[ \tilde{P}_1 \mid I_i \right] = \sigma^2 (1 - \rho_i).
\]

Note that \( \rho_i \) is not the correlation coefficient \( \frac{\text{cov}(\tilde{P}_1, \tilde{y}_i)}{\sigma_{\tilde{P}_1} \sigma_{\tilde{y}_i}} = \frac{\sigma^2}{\sigma \sqrt{\sigma^2 + \sigma^2_i}} = \sqrt{\rho_i} \).

Substituting these into (16.8), we have

\[
X_i = \frac{m + \rho_i (y_i - m) - R_f P_0}{a_i \sigma^2 (1 - \rho_i)}. \tag{16.10}
\]

From the denominator of (16.10) one sees that the individual’s demand for the risky asset is greater the lower is his risk aversion, \( a_i \), and the greater is the precision of his signal (the closer is \( \rho_i \) to 1, that is, the lower is \( \sigma_i \)). Now by aggregating the individual traders’ risky asset demands for shares and setting the sum equal to the fixed supply of shares, we can solve for the equilibrium risky asset price, \( P_0 \), that equates supply and demand:

\[
\bar{X} = \sum_{i=1}^{n} \left[ \frac{m + \rho_i (y_i - m) - R_f P_0}{a_i \sigma^2 (1 - \rho_i)} \right] \tag{16.11}
\]

\[
= \sum_{i=1}^{n} \left[ \frac{m + \rho_i (y_i - m)}{a_i \sigma^2 (1 - \rho_i)} \right] - \sum_{i=1}^{n} \left[ \frac{R_f P_0}{a_i \sigma^2 (1 - \rho_i)} \right]
\]
or

\[ P_0 = \frac{1}{R_f} \left[ \sum_{i=1}^{n} \frac{m + \rho_i (y_i - m)}{a_i \sigma^2 (1 - \rho_i)} - \bar{X} \right] \div \left[ \sum_{i=1}^{n} \frac{1}{a_i \sigma^2 (1 - \rho_i)} \right]. \]  (16.12)

From (16.12) we see that the price reflects a weighted average of the traders’ conditional expectation of the payoff of the risky asset. For example the weight on the \( i \)th trader’s conditional expectation, \( m + \rho_i (y_i - m) \), is

\[ \frac{1}{a_i \sigma^2 (1 - \rho_i)} \div \left[ \sum_{i=1}^{n} \frac{1}{a_i \sigma^2 (1 - \rho_i)} \right]. \]  (16.13)

The more precise (higher \( \rho_i \)) is trader \( i \)'s signal or the lower is his risk aversion (more aggressively he trades), the more that the equilibrium price reflects his expectations.

### 16.1.4 A Rational Expectations Equilibrium

The solution for the price, \( P_0 \), in equation (16.12) can be interpreted as a competitive equilibrium: each trader uses information on his own signal and, in equilibrium, takes the price of the risky asset as given in deciding on how much to demand of the risky asset. However, this equilibrium neglects the possibility of individual traders obtaining information about other traders’ signals from the equilibrium price itself, what practitioners call “price discovery.” In this sense, the previous equilibrium is not a rational expectations equilibrium. To see this, note that if traders initially formulate their demands according to equation (16.10) (using only information about their own signals), and the equilibrium price in (16.12) then results, an individual trader could infer information about the other traders’ signals from the formula for \( P_0 \) in (16.12). Hence, this trader would have the incentive to change his or her demand from that initially formulated in (16.10). This implies that equation (16.12) would not be the rational
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expectations equilibrium price.

Therefore, to derive a fully rational expectations equilibrium, we need to allow traders’ information sets to depend not only on their individual signals, but on the equilibrium price itself: \( I_i = \{ y_i, P^*_0(y) \} \) where \( y \equiv (y_1, y_2 \ldots y_n) \) is a vector of the traders’ individual signals.

In equilibrium, the aggregate demand for the shares of the risky asset must equal the aggregate supply, implying

\[
\bar{X} = \sum_{i=1}^{n} \left[ \frac{E \left[ \tilde{P}_1 \mid y_i, P^*_0(y) \right] - R_f P^*_0(y)}{a_i \text{Var} \left[ \tilde{P}_1 \mid y_i, P^*_0(y) \right]} \right].
\]

(16.14)

Now one can show that a rational expectations equilibrium exists for the case of the \( \epsilon_i \)'s being independent and having the same variance, that is, \( \sigma^2_i = \sigma^2 \), for \( i = 1, \ldots, n \).

**Theorem:** There exists a rational expectations equilibrium with \( P^*_0(y) \) given by

\[
P^*_0(y) = \frac{m + \rho (\bar{y} - m)}{R_f} - \frac{\sigma^2 (1 - \rho)}{R_f} \bar{X} \left/ \left[ \sum_{i=1}^{n} \frac{1}{a_i} \right] \right.
\]

(16.15)

where \( \bar{y} \equiv \frac{1}{n} \sum_{i=1}^{n} y_i \) and \( \rho \equiv \frac{\sigma^2}{\sigma^2 + \frac{n}{n}} \).

**Proof:** An intuitive outline of the proof is as follows.\(^1\) Note that in (16.15) \( P^*_0(y) \) is a linear function of \( \bar{y} \) with a fixed coefficient of \( \rho/R_f \). Therefore, if a trader observes \( P^*_0(y) \) (and knows the structure of the model, that is, the other parameters), then he can “invert” to infer the value of \( \bar{y} \). Now because all traders’ signals were assumed to have equal precision (same \( \sigma^2 \)), \( \bar{y} \), is a sufficient statistic for the information contained in all of the other signals. Further, because of the assumed independence of the signals, the precision of this average of signals is proportional to the number of traders, \( n \).

\(^1\)See the original Grossman article (Grossman 1976) for details.
Hence, the average signal would have the same precision as a single signal with variance $\frac{\sigma^2}{n}$.

Now if individual traders’ demands are given by equation (16.10) but where $y_i$ is replaced with $\bar{y}$ and $\rho_i$ is replaced with $\rho$, then by aggregating these demands and setting them equal to $\bar{X}$ as in equation (16.11), we end up with the solution in equation (16.15), which is consistent with our initial assumption that traders can invert $P_0^*(y)$ to find $\bar{y}$. Hence, $P_0^*(y)$ in equation (16.15) is the rational expectations equilibrium price of the risky asset.

Note that the information, $\bar{y}$, reflected in the equilibrium price is superior to any single trader’s private signal, $y_i$. In fact, since $\bar{y}$ is a sufficient statistic for all traders’ information, it makes knowledge of any single signal, $y_i$, redundant. The equilibrium would be the same if all traders received the same signal, $\bar{y} \sim N(0, \frac{\sigma^2}{n})$ or if they all decided to share information on their private signals among each other before trading commenced.

Therefore, the above equilibrium is a fully-revealing rational expectations equilibrium. The equilibrium price fully reveals all private information. This result has some interesting features in that it shows that prices can aggregate relevant information to help agents make more efficient investment decisions than would be the case if they relied solely on their private information and did not attempt to obtain information from the equilibrium price itself.

However, this fully revealing equilibrium is not robust to some small changes in assumptions. For example, suppose each trader needed to pay a tiny cost, $c$, to obtain his private signal, $y_i$. With any finite cost of obtaining information, the equilibrium would not exist because each individual receives no additional benefit from knowing $y_i$ given that they can observe $\bar{y}$ from the price. In other words, a given individual does not personally benefit from having private (inside) information in a fully-revealing equilibrium. In order for individuals to benefit
from obtaining (costly) information, we need an equilibrium where the price is only partially revealing. For this to happen, there needs to be one or more additional sources of uncertainty that add “noise” to individuals’ signals, so that other agents cannot infer them perfectly. We now turn to an example of a noisy rational expectations equilibrium.

16.1.5 A Noisy Rational Expectations Equilibrium

Let us make the following changes to the Grossman model’s assumptions along the lines of a model proposed by Bruce Grundy and Maureen McNichols (Grundy and McNichols 1989). Suppose that each trader begins the period with a random endowment of the risky asset. Specifically, trader $i$ possesses $\varepsilon_i$ shares of the risky asset so that her initial wealth is $W_0 = \varepsilon_i P_0$. The realization of $\varepsilon_i$ is known only to trader $i$. Across all traders, the endowments, $\varepsilon_i$, are independently and identically distributed with mean $\mu_X$ and variance $\sigma^2_X n$. To simplify the problem, we assume that the number of traders is very large. If we define $\bar{X}$ as the per capita supply of the risky asset and let $n$ go to infinity, then by the Central Limit Theorem $\bar{X}$ is a random variable distributed $N\left(\mu_X, \sigma^2_X\right)$. Note that in the limit as $n \to \infty$, the correlation between $\varepsilon_i$ and $\bar{X}$ becomes zero, so that trader $i$’s observation of her own endowment, $\varepsilon_i$, provides no information about the per capita supply, $\bar{X}$.

Next, let us modify the type of signal received by each trader to allow for a common error as well as a trader-specific error. Trader $i$ is assumed to receive the signal

$$\tilde{y}_i = \tilde{P}_1 + \tilde{\omega} + \tilde{\epsilon}_i$$

(16.16)

where $\tilde{\omega} \sim N(0, \sigma^2_{\omega})$ is the common error independent of $\tilde{P}_1$ and, as before, the idiosyncratic error $\tilde{\epsilon}_i \sim N(0, \sigma^2_{\epsilon_i})$ and is independent of $\tilde{P}_1$ and $\tilde{\omega}$. Because of
the infinite number of traders, it is realistic to allow for a common error so that traders, collectively, would not know the true payoff of the risky asset.

Recall from the Grossman model that the rational expectations equilibrium price in (16.15) was a linear function of \( \bar{y} \) and \( \bar{X} \). In the current model, the aggregate supply of the risky asset is not fixed, but random. However, this suggests that the equilibrium price will be of the form

\[
P_0 = \alpha_0 + \alpha_1 \bar{y} + \alpha_2 \bar{X} \tag{16.17}
\]

where now \( \bar{y} \equiv \lim_{n \to \infty} \sum^n_i y_i/n = \bar{P}_1 + \bar{\omega} \).

Although some assumptions differ, trader \( i \)'s demand for the risky asset continues to be of the form (16.8). Now recall that in a rational expectations equilibrium, investor \( i \)'s information set includes not only her private information but also the equilibrium price: \( I_i = \{y_i, P_0\} \). Given the assumed structure in (16.17) and the assumed normal distribution for \( \bar{P}_1, \bar{X}, \) and \( y_i \), then investor \( i \) optimally forecasts the end-of-period price as the projection

\[
E \left[ \bar{P}_1 | I_i \right] = \beta_0 + \beta_1 P_0 + \beta_2 y_i \tag{16.18}
\]

where

\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_0
\end{pmatrix} = \begin{pmatrix}
\alpha_1^2 (\sigma^2 + \sigma^2_\omega) + \alpha_2^2 \sigma^2_X & \alpha_1 (\sigma^2 + \sigma^2_\omega) \\
\alpha_1 (\sigma^2 + \sigma^2_\omega) & \sigma^2 + \sigma^2_\omega + \sigma^2_X
\end{pmatrix}^{-1} \begin{pmatrix}
\alpha_1^2 \sigma^2 \\
\sigma^2
\end{pmatrix} - \begin{pmatrix}
\alpha_0 - \alpha_1 m - \alpha_2 \mu_X \\
- \beta_1 (\alpha_0 - \alpha_1 m - \alpha_2 \mu_X) - \beta_2 m
\end{pmatrix} \tag{16.19}
\]
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If we then average the $X_i$ in (16.8) over all investors, one obtains

\[
X = \frac{\beta_0 + (\beta_1 - R_f) P_0 + \beta_2 \bar{y}}{\bar{\text{Var}} \left[ \hat{P}_1 \mid I_i \right]} \tag{16.20}
\]

\[
= \frac{\beta_0}{\bar{\text{Var}} \left[ \hat{P}_1 \mid I_i \right]} + \frac{\beta_1 - R_f}{\bar{\text{Var}} \left[ \hat{P}_1 \mid I_i \right]} P_0 + \frac{\beta_2}{\bar{\text{Var}} \left[ \hat{P}_1 \mid I_i \right]} \bar{y}
\]

where $\bar{\pi} \equiv 1/ \left( \lim_{n \to \infty} \frac{1}{n} \sum_i \frac{1}{\pi_i} \right)$ is the harmonic mean of investors’ risk aversions. Now note that we can re-write equation (16.17) as

\[
X = -\frac{\alpha_0}{\alpha_2} + \frac{1}{\alpha_2} P_0 - \frac{\alpha_1}{\alpha_2} \bar{y} \tag{16.21}
\]

In a rational expectations equilibrium, the relationships between the variables $X$, $P_0$, and $\bar{y}$ must be consistent with the individual investors’ expectations. This implies that the intercepts, and the coefficients on $P_0$ and on $\bar{y}$, must be identical in equations (16.20) and (16.21). By matching the intercepts and coefficients, we obtain three nonlinear equations in the three unknowns $\alpha_0$, $\alpha_1$, and $\alpha_2$. Although explicit solutions for $\alpha_0$, $\alpha_1$, and $\alpha_2$ cannot be obtained, we can still interpret some of the characteristics of the equilibrium. To see this, note that if the coefficients on $\bar{y}$ are equated, one obtains

\[
-\frac{\alpha_1}{\alpha_2} = \frac{\beta_2}{\bar{\text{Var}} \left[ \hat{P}_1 \mid I_i \right]} \tag{16.22}
\]

Using (16.19) to substitute for $\beta_2$ and the variance of the projection of $\hat{P}_1$ on $I_i$ to substitute for $\text{Var} \left[ \hat{P}_1 \mid I_i \right]$, (16.22) can be re-written as

\[
-\frac{\alpha_1}{\alpha_2} = \frac{\sigma_X^2}{\bar{\pi} \left[ \sigma_X^2 (\sigma_\omega^2 + \sigma_\epsilon^2) + (\alpha_1/\alpha_2)^2 \sigma_\omega^2 \sigma_\epsilon^2 \right]} \tag{16.23}
\]

This is a cubic equation in $\alpha_1/\alpha_2$. The ratio $\alpha_1/\alpha_2$ is a measure of how
aggressively an individual investor responds to his individual private signal, relative to the average signal, $\bar{y}$, reflected in $P_0$. To see this, note that if one uses (16.8), (16.20), (16.21), and $y_i - \bar{y} = \tilde{\epsilon}_i$, the individual’s demand for the risky asset can be written as

$$X_i = \frac{\pi}{a_i} \left( X - \frac{\alpha_1}{\alpha_2} \tilde{\epsilon}_i \right) \quad (16.24)$$

From (16.24) one sees that if there were no information differences, each investor would demand a share of the average supply of the risky asset, $X$, in proportion to the ratio of the harmonic average of risk aversions to his own risk aversion. However, unlike the fully revealing equilibrium of the previous section, the individual investor cannot perfectly invert the equilibrium price to find the average signal in (16.17) due to the uncertain aggregate supply shift, $X$. Hence, individual demands do respond to private information as reflected by $\tilde{\epsilon}_i$. The ratio $\alpha_1/\alpha_2$ reflects the simultaneous equation problem faced by the investor in trying to sort out a shift in supply, $X$, from a shift in aggregate demand generated by $\bar{y}$. From (16.23) we see that as $\sigma_\omega^2 \to \infty$ or $\sigma_\epsilon^2 \to \infty$, so that investors’ private signals become uninformative, then $\alpha_1/\alpha_2 \to 0$ and private information has no effect on demands or the equilibrium price. If, instead, $\sigma_\omega^2 = 0$, so that there is no common error, then (16.23) simplifies to

$$-\frac{\alpha_1}{\alpha_2} = \frac{1}{a_i \sigma_\epsilon^2} \quad (16.25)$$

and (16.24) becomes

$$X_i = \frac{\pi}{a_i} X - \frac{1}{a_i \sigma_\epsilon^2} \tilde{\epsilon}_i \quad (16.26)$$

so that an individual’s demand responds to her private signal in direct proportion to the signal’s precision and indirect proportion to her risk aversion.
16.2 Asymmetric Information, Trading, and Markets

Let us now consider another model with private information that is pertinent to a security market organized by a market maker. This market maker, who might be thought of as a specialist on a stock exchange or a security dealer in an over-the-counter market, sets a risky asset’s price with the recognition that he may be trading at that price with a possibly better-informed individual. Albert "Pete" Kyle (Kyle 1985) developed this model, and it has been widely applied to study market micro-structure issues. The model is similar to the previous one in that the equilibrium security price partially reveals the better-informed individual’s private information. Also like the previous model, there is an additional source of uncertainty that prevents a fully-revealing equilibrium, namely, orders from uninformed "noise" or "liquidity" traders that provide camouflage for the better-informed individual’s insider trades. The model’s results provide insights regarding the factors affecting bid-ask spreads and the market impact of trades.

16.2.1 Assumptions of the Kyle Model

The Kyle model is based on the following assumptions.

A.1 Asset Return Distribution

The model is a single period model. At the beginning of the period, agents trade in an asset that has a random end of period liquidation value of \( \tilde{\nu} \sim N(p_0, \Sigma_0) \).

A.2 Liquidity Traders

Pete Kyle’s paper (Kyle 1985) also contains a multi-period continuous time version of his single period model. Jiang Wang (Wang 1993) has also constructed a continuous time asset pricing model with asymmetrically-informed investors who have constant absolute risk aversion utility.
Noise traders have needs to trade that are exogenous to the model. It is assumed that they, as a group, submit a “market” order to buy \( \tilde{u} \) shares of the asset, where \( \tilde{u} \sim N(0, \sigma_u^2) \). \( \tilde{u} \) and \( \tilde{\nu} \) are assumed to be independently distributed.\(^3\)

A.3 Better-Informed Traders

The single risk-neutral insider is assumed have better information than the other agents. He knows with perfect certainty the realized end of period value of the risky security \( \tilde{\nu} \) (but not \( \tilde{u} \)) and chooses to submit a market order of size \( x \) that maximizes his expected end of period profits.\(^4\)

A.4 Competitive Market Maker

The single risk-neutral market maker (for example, a New York Stock Exchange specialist) receives the market orders submitted by the noise traders and the insider, which in total equal \( u + x \). Importantly, the market maker cannot distinguish what part of this total order consists of orders made by noise traders and what part consists of the order of the insider. (The traders are anonymous.) The market maker sets the market price, \( p \), and then takes the position \( -(u + x) \) to clear the market. It is assumed that market making is a perfectly competitive profession, so that the market maker sets the price \( p \) such that, given the total order submitted, his profit at the end of the period is expected to be zero.

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\(^3\)Why rational noise traders submit these orders has been modeled by assuming they have exogenous shocks to their wealth and need to rebalance their portfolio (Spiegel and Subrahmanyam 1992) or by assuming that they have uncertainty regarding the timing of their consumption (Gorton and Pennacchi 1993).

\(^4\)This assumption can be weakened to the case of the insider having uncertainty over \( \tilde{\nu} \) but having more information on \( \tilde{\nu} \) than the other traders. One can also allow the insider to submit “limit” orders, that is, orders that are a function of the equilibrium market price (a demand schedule), as in another model by Pete Kyle (Kyle 1989).
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16.2.2 Trading and Pricing Strategies

Since the noise traders’ order is exogenous, we need only to consider the optimal actions of the market maker and the insider.

The market maker observes only the total order flow, $u + x$. Given this information, he must then set the equilibrium market price $p$ that gives him zero expected profits. Since his end of period profits are $-(\bar{\nu} - p)(u + x)$, this implies that the price set by the market maker satisfies

$$p = E[\bar{\nu} | u + x] \quad (16.27)$$

The information on the total order size is important to the market maker. The greater the total order size, the more likely it is that $x$ is large due to the insider knowing that $\nu$ is greater than $p_0$. Thus, the market maker would tend to set $p$ higher than otherwise. Similarly, if $u + x$ is low, the more likely it is that $x$ is low because the insider knows $\nu$ is below $p_0$ and is submitting a sell order. In this case, the market maker would tend to set $p$ lower than otherwise. Thus, the pricing rule of the market maker is a function of $x + u$, that is, $P(x + u)$.

Since the insider sets $x$, it is an endogenous variable that depends on $\bar{\nu}$. The insider chooses $x$ to maximize his expected end of period profits, $\tilde{\pi}$, given knowledge of $\nu$ and the way that the market maker behaves in setting the equilibrium price:

$$\max_x E[\tilde{\pi} | \nu] = \max_x E[(\nu - P(x + \bar{u}))x | \nu] \quad (16.28)$$

An equilibrium in this model is a pricing rule chosen by the market maker and a trading strategy chosen by the insider such that: the insider maximizes expected profits, given the market maker’s pricing rule; the market maker sets the price to earn zero expected profits, given the trading strategy of the insider;
the insider and market maker have rational expectations, that is, the equilibrium is a fixed-point where an agent’s actual behavior is that expected by the other.

**Insider’s Trading Strategy**

Suppose the market maker chooses a market price that is a linear function of the total order flow, \( P(y) = \mu + \lambda y \). We will later argue that a linear pricing rule is optimal. If this is so, what is the insider’s choice of \( x \)? From (16.28) we have

\[
\max_x E[(\nu - P(x + \tilde{u}))x \mid \nu] = \max_x E[(\nu - \mu - \lambda (x + \tilde{u}))x \mid \nu] \tag{16.29}
\]

\[
= \max_x (\nu - \mu - \lambda x)x, \text{ since } E[\tilde{u}] = 0
\]

Thus, the solution to the insider’s problem in (16.29) is

\[
x = \alpha + \beta \nu \tag{16.30}
\]

where \( \alpha = -\frac{\mu}{2\lambda} \) and \( \beta = \frac{1}{2\lambda} \). Therefore, if the market maker uses a linear pricing-setting rule, the optimal trading strategy for the insider is a linear trading rule.

**Market Maker’s Pricing Strategy**

Next, let us return to the market maker’s problem of choosing the market price that, conditional on knowing the total order flow, results in a competitive (zero) expected profit. Given the assumption that market making is a perfectly competitive profession, a market maker needs to choose the “best” possible estimate of \( E[\nu \mid u + x] \) in setting the price \( p = E[\nu \mid u + x] \). What estimate of the mean of \( \nu \) is best? The maximum likelihood estimate of \( E[\nu \mid u + x] \) is best in the sense that it attains maximum efficiency and is also the minimum variance un-
biased estimate.

Note that if the insider follows the optimal trading strategy, which according to equation (16.30) is \( x = \alpha + \beta \tilde{\nu} \), then from the point of view of the market maker, \( \tilde{\nu} \) and \( y \equiv \tilde{u} + x = \tilde{u} + \alpha + \beta \tilde{\nu} \) are jointly normally distributed. Because \( \nu \) and \( y \) are jointly normal, the maximum likelihood estimate of the mean of \( \nu \) conditional on \( y \) is linear in \( y \), that is, \( E[\tilde{\nu} \mid y] \) is linear in \( y \) when they are jointly normally distributed. Hence, the previously assumed linear pricing rule is, in fact, optimal in equilibrium. Therefore, the market maker should use the maximum likelihood estimator, which in the case of \( \nu \) and \( y \) being normally distributed is equivalent to the “least squares” estimator. This is the one that minimizes

\[
E \left[ (\tilde{\nu} - P(y))^2 \right] = E \left[ (\tilde{\nu} - \mu - \lambda y)^2 \right] 
= E \left[ (\tilde{\nu} - \mu - \lambda (\tilde{u} + \alpha + \beta \tilde{\nu}))^2 \right] 
\]  

Thus, the optimal pricing rule equals \( \mu + \lambda y \) where \( \mu \) and \( \lambda \) minimize

\[
\min_{\mu, \lambda} E \left[ (\tilde{\nu} (1 - \lambda \beta) - \lambda \tilde{u} - \mu - \lambda \alpha)^2 \right] 
\] (16.32)

Recalling the assumptions \( E[\nu] = p_0 \), \( E[(\nu - p_0)^2] = \Sigma_0 \), \( E[u] = 0 \), \( E[u^2] = \sigma_u^2 \), and \( E[\nu u] = 0 \), the above objective function can be written as

\[
\min_{\mu, \lambda} (1 - \lambda \beta)^2 (\Sigma_0 + p_0^2) + (\mu + \lambda \alpha)^2 + \lambda^2 \sigma_u^2 - 2(\mu + \lambda \alpha) (1 - \lambda \beta) p_0 
\] (16.33)

The first order conditions with respect to \( \mu \) and \( \lambda \) are
\[ \mu = -\lambda \alpha + p_0 (1 - \lambda \beta) \]  

(16.34)

\[ 0 = -2\beta (1 - \lambda \beta) (\Sigma_0 + p_0^2) + 2\alpha (\mu + \lambda \alpha) + 2\lambda \sigma_u^2 \]

\[-2p_0 [-\beta (\mu + \lambda \alpha) + \alpha (1 - \lambda \beta)] \]  

(16.35)

Substituting \( \mu + \lambda \alpha = p_0 (1 - \lambda \beta) \) from (16.34) into (16.35), we see that (16.35) simplifies to

\[ \lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} \]  

(16.36)

Substituting in for the definitions \( \alpha = -\frac{\mu}{2\lambda} \) and \( \beta = \frac{1}{2\lambda} \) in (16.34) and (16.35), we have

\[ \mu = p_0 \]  

(16.37)

\[ \lambda = \frac{1}{2} \frac{\sqrt{\Sigma_0}}{\sigma_u} \]  

(16.38)

In summary, the equilibrium price is

\[ p = p_0 + \frac{1}{2} \frac{\sqrt{\Sigma_0}}{\sigma_u} (\tilde{u} + \tilde{x}) \]  

(16.39)

where the equilibrium order submitted by the insider is

\[ x = \frac{\sigma_u}{\sqrt{\Sigma_0}} (\tilde{v} - p_0) \]  

(16.40)

16.2.3 Analysis of the Results
From (16.40), we see that the greater is the volatility (amount) of noise trading, \( \sigma_u \), the larger is the magnitude of the order submitted by the insider for a given deviation of \( \nu \) from its unconditional mean. Hence, the insider trades more actively on his private information the greater is the “camouflage” provided by noise trading. Greater noise trading makes it more difficult for the market maker to extract the “signal” of insider trading from the noise. Note that if equation (16.40) is substituted into (16.39), one obtains

\[
p = p_0 + \frac{1}{2} \frac{\sqrt{\Sigma_0}}{\sigma_u} \tilde{u} + \frac{1}{2} (\tilde{\nu} - p_0)
\]

(16.41)

Thus we see that only one-half of the insider’s private information, \( \frac{1}{2} \tilde{\nu} \), is reflected in the equilibrium price, so that the price is not fully revealing.\(^5\) To obtain an equilibrium of incomplete revelation of private information, it was necessary to have a second source of uncertainty, namely, the amount of noise trading.

Using (16.40) and (16.41), we can calculate the insider’s expected profits:

\[
E[\tilde{\pi}] = E[x(\nu - p)] = E \left[ \frac{\sigma_u}{\sqrt{\Sigma_0}} (\tilde{\nu} - p_0) \frac{1}{2} \left( \nu - p_0 - \frac{\sqrt{\Sigma_0}}{\sigma_u} \tilde{u} \right) \right]
\]

(16.42)

Conditional on knowing \( \nu \), that is, after learning the realization of \( \nu \) at the beginning of the period, the insider expects profits of

\[
E[\tilde{\pi} | \nu] = \frac{1}{2} \frac{\sigma_u}{\sqrt{\Sigma_0}} (\nu - p_0)^2
\]

(16.43)

\(^5\)A fully-revealing price would be \( p = \tilde{\nu} \).
Hence, the larger is $\nu$’s deviation from $p_0$, the larger the expected profit. Unconditional on knowing $\tilde{\nu}$, that is, before the start of the period, the insider expects a profit of

$$E[\tilde{\pi}] = \frac{1}{2} \frac{\sigma_u}{\sqrt{\Sigma_0}} E[\tilde{\nu} - p_0]^2 = \frac{1}{2} \sigma_u \sqrt{\Sigma_0}$$

(16.44)

which is proportional to the standard deviations of noise traders’ order and the end of period value of $\nu$.

Since, by assumption, the market maker sets the security price in a way that gives him zero expected profits, the expected profits of the insider equals the expected losses of the noise traders. In other words, it is the noise traders, not the market maker, that lose, on average, from the presence of the insider.

From equation (16.39), we see that $\lambda = \frac{1}{2} \frac{\sqrt{\Sigma_0}}{\sigma_u}$ is the amount that the market maker raises the price when the total order flow, $(u + x)$, goes up by 1 unit. This can be thought of as relating to the security’s bid-ask spread, that is, the difference the price for sell orders versus buy orders, though here sell and buy prices are not fixed but are a function of the order size since the pricing rule is linear. Moreover, since the amount of order flow necessary to raise the price by $\$$1 equals $1/\lambda = 2 \frac{\sigma_u}{\sqrt{\Sigma_0}}$, the model provides a measure of the “depth” of the market or market “liquidity.” The higher is the proportion of noise trading to the value of insider information, $\frac{\sigma_u}{\sqrt{\Sigma_0}}$, the deeper or more liquid is the market.

Intuitively, the more noise traders relative to the value of insider information, the less the market maker needs to adjust the price in response to a given order, since the likelihood of the order being that of a noise trader, rather than an insider, is greater. While the greater is the number of noise traders (that is, the greater is $\sigma_u$), the greater is the expected profits of the insider (see equation (16.44)) and the greater is the total expected losses of the noise traders. However, the expected loss per individual noise trader falls with the
greater level of noise trading.\(^6\)

### 16.3 Summary

The models considered in this chapter analyze the degree to which private information about an asset’s future payoff or value is reflected in the asset’s current price. An investor’s private information affects an asset’s price by determining the investor’s desired demand (long or short position) for the asset, though the investor’s demand also is tempered by risk-aversion. More subtly, we saw that a rational investor can also learn about the private information of other investors through the asset’s price itself, and this price discovery affects the investors’ equilibrium demands. Indeed, under some circumstances, the asset’s price may fully reveal all relevant private information such that any individual’s private information becomes redundant.

Perhaps more realistically, there are non-information-based factors that affect the net supply or demand for an asset. These "noise" factors prevent investors from perfectly inferring the private information signals of others, resulting in an asset price that is less than fully revealing. Noise provides camouflage for investors with private information, allowing these traders to profit from possessing such information. Their profits come at the expense of liquidity traders since the greater the likelihood of private information regarding a security, the larger will be the security’s bid-ask spread. Hence, this theory predicts that a security’s liquidity is determined by the degree of noise (non-information-based) trading relative to insider (private-information-based) trading.

### 16.4 Exercises

\(^6\)Gary Gorton and George Pennacchi (Gorton and Pennacchi 1993) derive this result by modeling individual liquidity traders.
1. Show that the maximization problem in (16.7) is equivalent to the maximization problem in (16.5).

2. Consider a special case of the Grossman model. Traders can choose between holding a risk-free asset, which pays an end-of-period return of $R_f$, or a risky asset that has a beginning of period price of $P_0$ per share and has an end-of-period payoff (price) of $P_1$ per share. The unconditional distribution of $P_1$ is assumed to be $N(m, \sigma^2)$. The risky asset is assumed to be a derivative security, such as a futures contract, so that its net supply equals zero.

There are two different traders who maximize expected utility over end-of-period wealth, $\tilde{W}_{1i}$, $i = 1, 2$. The form of the $i^{th}$ trader’s utility function is

$$U_i(\tilde{W}_{1i}) = -e^{-a_i\tilde{W}_{1i}}, \ a_i > 0.$$ 

At the beginning of the period, the $i^{th}$ trader observes $y_i$ which is a noisy signal of the end-of-period value of the risky asset

$$y_i = \tilde{P}_1 + \tilde{\epsilon}_i$$

where $\epsilon_i \sim N(0, \sigma^2_i)$ and is independent of $\tilde{P}_1$. Note that the variances of the traders’ signals are the same. Also assume $E[\epsilon_1\epsilon_2] = 0$.

2.a Suppose each trader does not attempt to infer the other trader’s information from the equilibrium price, $P_0$. Solve for each of the traders’ demands for the risky asset and the equilibrium price, $P_0$. 


2.b Now suppose each trader does attempt to infer the other’s signal from the equilibrium price, \( P_0 \). What will be the rational expectations equilibrium price in this situation? What will be each of the traders’ equilibrium demands for the risky asset?

3. In the Kyle (1985) model, replace the original assumption A.3 with the following new one:

**A.3 Better-Informed Traders**

The single risk-neutral insider is assumed have better information than the other agents. He observes a signal of the asset’s end-of-period value equal to

\[
s = \tilde{v} + \tilde{\epsilon}
\]

where \( \tilde{\epsilon} \sim N(0, \sigma^2_\epsilon) \), \( 0 < \sigma^2_\epsilon < \Sigma_0 \), and \( \tilde{\epsilon} \) is distributed independently of \( \tilde{u} \) and \( \tilde{\nu} \). The insider does not observe \( \tilde{u} \), but chooses to submit a market order of size \( x \) that maximizes his expected end-of-period profits.

3.a Suppose that the market maker’s optimal price setting rule is a linear function of the order flow

\[
p = \mu + \lambda (u + x) .
\]

Write down the expression for the insider’s expected profits given this pricing rule.

3.b Take the first order condition with respect to \( x \) and solve for the insider’s optimal trading strategy as a function of the signal and the parameters of the market maker’s pricing rule.